[C—503]

A KEY

TO

ALGEBRA

PART

BY

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PREFACE.

THIS Book is published at the request of many Teachers. It is intended to assist, *first*, Masters who cannot devote much time to the examination of the work of their pupils, and *secondly*, Students who are unable to obtain the help of competent instructors. In working the Exercises, the simplest and most obvious methods of solution have in all cases been adopted.

J. HAMBLIN SMITH.

TABLE OF REFERENCE TO EXERCISES IN ELEMENTARY ALGEBRA.

ELEMENTARY ALGEBRA.

KEY.

IV.

1. $a+b+3a-2b=4a-b.$

2. $a+b-a+3b=4b.$

3. $3a+5b-6c-2a-4b+2c=a+b-4c.$

4. $a+b-c-a+b+c=2b.$

5. $14x-5x+9-\{4-3x-2x+3\}=14x-5x+9-4+3x+2x-3=14x+2.$

6. $4x-\{3x-(2x-x+a)\}=4x-\{3x-2x+x-a\}=4x-3x+2x-x+a$
$=2x+a.$

7. $15x-\{7x+(3x+a-x)\}=15x-\{7x+3x+a-x\}$
$=15x-7x-3x-a+x=6x-a.$

8. $a-[b+\{a-b-a\}]=a-[b+a-b-a]=a-b-a+b+a=a.$

9. $6a+[4a-\{8b-2a-4b-22b\}-7b]-[7b+\{8a-3b-4a+8b\}+6a]$
$=6a+[4a-8b+2a+4b+22b-7b]-[7b+8a-3b-4a+8b+6a]$
$=6a+4a-8b+2a+4b+22b-7b-7b-8a+3b+4a-8b-6a$
$=2a-b.$

10. $b-[b-a-b-\{b-(b-a+b)\}]=b-[b-a-b-\{b-b+a-b\}]$
$=b-[b-a-b-b+b-a+b]=b-b+a+b+b-b+a-b=2a.$

11. $2c-6a+b-\{c-5a-2b-a+3b\}=2c-6a+b-c+5a+2b+a-3b=c.$

12. $2x-\{a-(2a-[3a-(4a-[5a-6a+x])])\}$
$=2x-\{a-(2a-[3a-(4a-5a+6a-x)])\}$
$=2x-\{a-(2a-[3a-4a+5a-6a+x])\}$
$=2x-\{a-(2a-3a+4a-5a+6a-x)]\}$
$=2x-\{a-2a+3a-4a+5a-6a+x\}$
$=2x-a+2a-3a+4a-5a+6a-x=x+3a.$

13. $25a-19b-[3b-\{4a-5b+6c\}]=25a-19b-[3b-4a+5b-6c]$
$=25a-19b-3b+4a-5b+6c=29a-27b+6c.$

VIII.

1.
$$\begin{array}{l} x+3 \\ x+9 \\ \hline x^2+3x \\ \quad\;\; +9x+27 \\ \hline x^2+12x+27 \end{array}$$

2.
$$\begin{array}{l} x+15 \\ x-7 \\ \hline x^2+15x \\ \quad\;\; -7x-105 \\ \hline x^2+8x-105 \end{array}$$

3.
$$\begin{array}{l} x-12 \\ x+10 \\ \hline x^2-12x \\ \quad\;\; +10x-120 \\ \hline x^2-2x-120 \end{array}$$

4.
$$\begin{array}{l} x-8 \\ x-7 \\ \hline x^2-8x \\ \quad\;\; -7x+56 \\ \hline x^2-15x+56 \end{array}$$

5.
$$\begin{array}{l} a-3 \\ a-5 \\ \hline a^2-3a \\ \quad\;\; -5a+15 \\ \hline a^2-8a+15 \end{array}$$

6.
$$\begin{array}{l} y-6 \\ y+13 \\ \hline y^2-6y \\ \quad\;\; +13y-78 \\ \hline y^2+7y-78 \end{array}$$

7.
$$\begin{array}{l} x^2-4 \\ x^2+5 \\ \hline x^4-4x^2 \\ \quad\;\; +5x^2-20 \\ \hline x^4+x^2-20 \end{array}$$

8.
$$\begin{array}{l} x^2-6x+9 \\ x^2-6x+5 \\ \hline x^4-6x^3+9x^2 \\ \quad\;\; -6x^3+36x^2-54x \\ \qquad\qquad\quad +5x^2-30x+45 \\ \hline x^4-12x^3+50x^2-84x+45 \end{array}$$

9.
$$\begin{array}{l} x^2+5x-3 \\ x^2-5x-3 \\ \hline x^4+5x^3-3x^2 \\ \quad -5x^3-25x^2+15x \\ \qquad\quad -3x^2-15x+9 \\ \hline x^4 \qquad -31x^2 \qquad +9 \end{array}$$

10.
$$\begin{array}{l} a^3-3a+2 \\ a^3-3a^2+2 \\ \hline a^6-3a^4+2a^3 \\ \quad -3a^5+9a^3-6a^2 \\ \qquad\quad +2a^3-6a+4 \\ \hline a^6-3a^5-3a^4+13a^3-6a^2-6a+4 \end{array}$$

11.
$$\begin{array}{l} x^2-x+1 \\ x^2+x-1 \\ \hline x^4-x^3+x^2 \\ \quad +x^3-x^2+x \\ \qquad\quad -x^2+x-1 \\ \hline x^4 \qquad -x^2+2x-1 \end{array}$$

12.
$$\begin{array}{l} x^2+xy+y^2 \\ x^2-xy+y^2 \\ \hline x^4+x^3y+x^2y^2 \\ \quad -x^3y-x^2y^2-xy^3 \\ \qquad\quad +x^2y^2+xy^3+y^4 \\ \hline x^4 \qquad +x^2y^2 \qquad +y^4 \end{array}$$

13.
$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \qquad\qquad -y^3 \end{array}$$

14.
$$\begin{array}{l} a^2-x^2 \\ a^4+a^2x^2+x^4 \\ \hline a^6-a^4x^2 \\ \quad +a^4x^2-a^2x^4 \\ \qquad\quad +a^2x^4-x^6 \\ \hline a^6 \qquad\qquad -x^6 \end{array}$$

15.
$$\begin{array}{l} x^3-3x^2+3x-1 \\ x^2+3x+1 \\ \hline x^5-3x^4+3x^3-x^2 \\ \quad +3x^4-9x^3+9x^2-3x \\ \qquad\quad +x^3-3x^2+3x-1 \\ \hline x^5 \qquad -5x^3+5x^2 \qquad -1 \end{array}$$

16.
$$\begin{array}{l} x^3+3x^2y+9xy^2+27y^3 \\ x-3y \\ \hline x^4+3x^3y+9x^2y^2+27xy^3 \\ \quad -3x^3y-9x^2y^2 \;\; 27xy^3-81y^4 \\ \hline x^4 \qquad\qquad\qquad -81y^4 \end{array}$$

17.
$$\begin{array}{l} a^3+2a^2b+4ab^2+8b^3 \\ a-2b \\ \hline a^4+2a^3b+4a^2b^2+8ab^3 \\ \quad -2a^3b-4a^2b^2-8ab^3-16b^4 \\ \hline a^4 \qquad\qquad\qquad -16b^4 \end{array}$$

18.
$$\begin{array}{l} 8a^3+4a^2b+2ab^2+b^3 \\ 2a-b \\ \hline 16a^4+8a^3b+4a^2b^2+2ab^3 \\ \quad -8a^3b-4a^2b^2-2ab^3-b^4 \\ \hline 16a^4 \qquad\qquad\qquad -b^4 \end{array}$$

19.
$$\begin{array}{l}
a^3-2a^2b+3ab^2+4b^3\\
a^2-2ab-3b^2\\
\hline
a^5-2a^4b+3a^3b^2+4a^2b^3\\
\quad -2a^4b+4a^3b^2-6a^2b^3-8ab^4\\
\qquad\qquad -3a^3b^2+6a^2b^3-9ab^4-12b^5\\
\hline
a^5-4a^4b+4a^3b^2+4a^2b^3-17ab^4-12b^5
\end{array}$$

20.
$$\begin{array}{l}
a^3+3a^2b-2ab^2+3b^3\\
a^2+2ab-3b^2\\
\hline
a^5+3a^4b-2a^3b^2+3a^2b^3\\
\quad +2a^4b+6a^3b^2-4a^2b^3+6ab^4\\
\qquad\qquad -3a^3b^2-9a^2b^3+6ab^4-9b^5\\
\hline
a^5+5a^4b+a^3b^2-10a^2b^3+12ab^4-9b^5
\end{array}$$

21.
$$\begin{array}{l}
a^2-2ax+4x^2\\
a^2+2ax+4x^2\\
\hline
a^4-2a^3x+4a^2x^2\\
\quad +2a^3x-4a^2x^2+8ax^3\\
\qquad\qquad +4a^2x^2-8ax^3+16x^4\\
\hline
a^4 \qquad +4a^2x^2 \qquad +16x^4
\end{array}$$

22.
$$\begin{array}{l}
9a^2+3ax+x^2\\
9a^2-3ax+x^2\\
\hline
81a^4+27a^3x+9a^2x^2\\
\quad -27a^3x-9a^2x^2-3ax^3\\
\qquad\qquad +9a^2x^2+3ax^3+x^4\\
\hline
81a^4 \qquad +9a^2x^2 \qquad +x^4
\end{array}$$

23.
$$\begin{array}{l}
x^4-2ax^2+4a^2\\
x^4+2ax^2+4a^2\\
\hline
x^8-2ax^6+4a^2x^4\\
\quad +2ax^6-4a^2x^4+8a^3x^2\\
\qquad\qquad +4a^2x^4-8a^3x^2+16a^4\\
\hline
x^8 \qquad +4a^2x^4 \qquad +16a^4
\end{array}$$

24.
$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc\\
a\ +b\ +c\\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc\\
\quad +a^2b+b^3\ +bc^2\ -ab^2-abc-b^2c\\
\qquad\qquad +a^2c+b^2c+c^3-abc-ac^2-bc^2\\
\hline
a^3+b^3\ +c^3\ -3abc
\end{array}$$

25. $$\begin{array}{l} x^2+4xy+5y^2 \\ x^3-3x^2y-2xy^2+3y^3 \\ \hline x^5+4x^4y+5x^3y^2 \\ \quad -3x^4y-12x^3y^2-15x^2y^3 \\ \qquad -2x^3y^2-8x^2y^3-10xy^4 \\ \qquad\qquad +3x^2y^3+12xy^4+15y^5 \\ \hline x^5+x^4y-9x^3y^2-20x^2y^3+2xy^4+15y^5 \end{array}$$

26. $$\begin{array}{l} ab+cd+ac+bd \\ ab+cd-ac-bd \\ \hline a^2b^2+abcd+a^2bc+ab^2d \\ \quad +abcd+c^2d^2+ac^2d+bcd^2 \\ \qquad -a^2bc-ac^2d-a^2c^2-abcd \\ \qquad\qquad -ab^2d-bcd^2-abcd-b^2d^2 \\ \hline a^2b^2 \qquad +c^2d^2 \qquad -a^2c^2 \qquad -b^2d^2 \end{array}$$

27. $(x-a)(x+a)(x^2+a^2)(x^4+a^4)=(x^2-a^2)(x^2+a^2)(x^4+a^4)$
$=(x^4-a^4)(x^4+a^4)=x^8-a^8$.

28. $(x-a)(x+b)(x-c)=(x^2-ax+bx-ab)(x-c)$
$=x^3-ax^2+bx^2-cx^2-abx+acx-bcx+abc$.

29. $(1-x)(1+x)(1+x^2)(1+x^4)=(1-x^2)(1+x^2)(1+x^4)$
$=(1-x^4)(1+x^4)=1-x^8$.

30. First multiply x^2+xy+y^2 by $x-y$; the result is x^3-y^3.
Then multiply x^2-xy+y^2 by $x+y$; the result is x^3+y^3.
Then multiply x^3-y^3 by x^3+y^3; the result is x^6-y^6.

31. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$=(a^2-x^2)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$=(a^4-x^4)(a^4+x^4)(a^8+x^8)$
$=(a^8-x^8)(a^8+x^8)=a^{16}-x^{16}$.

32. $(x-5)(x-6)(x+7)=(x^2-11x+30)(x+7)=x^3-4x^2-47x+210$;
therefore the coefficient of x is -47.

33. $(x+8)(x+3)(x-2)=(x^2+11x+24)(x-2)=x^3+9x^2+2x-48$;
therefore the coefficient of x is 2.

34. $(x-2)(x-3)(x+4)=(x^2-5x+6)(x+4)=x^3-x^2-14x+24$;
therefore the coefficient of x is -14.

35. $(x-a)(x-b)(x-c)=(x^2-ax-bx+ab)(x-c)$
$=x^3-ax^2-bx^2-cx^2+abx+acx+bcx-abc$;
therefore the coefficient of x is $ab+ac+bc$.

36. $(x^2+3x-2)(x^2-3x+2)(x^4-5)=(x^4-9x^2+12x-4)(x^4-5)$
$=x^8-9x^6+12x^5-9x^4+45x^2-60x+20$;
therefore the coefficient of x is -60.

37. $(x^2-x+1)(x^2+x-1)(x^4-x^2+1)=(x^4-x^2+2x-1)(x^4-x^2+1)$
$=x^8-2x^6+2x^5+x^4-2x^3+2x-1$;
therefore the coefficient of x is 2.

38. $(x^2-mx+1)(x^2-mx-1)(x^4-m^2x-1)$
$=(x^4-2mx^3+m^2x^2-1)(x^4-m^2x-1)=x^8-2mx^7+m^2x^6-m^2x^5$
$+2m^3x^4-2x^4-m^4x^3+2mx^3-m^2x^2+m^2x+1$;
therefore the coefficient of x is m^2.

IX.

8.
$$\begin{array}{l}
-a^3-a^2-a \\
-a-1 \\
\hline
a^4+a^3+a^2 \\
\quad +a^3+a^2+a \\
\hline
a^4+2a^3+2a^2+a
\end{array}$$

9.
$$\begin{array}{l}
3x^2y-5xy^2+4y^3 \\
-2x-3y \\
\hline
-6x^3y+10x^2y^2-8xy^3 \\
\qquad -9x^2y^2+15xy^3-12y^4 \\
\hline
-6x^3y+x^2y^2+7xy^3-12y^4
\end{array}$$

10.
$$\begin{array}{l}
-5m^2-6mn+7n^2 \\
-m+n \\
\hline
5m^3+6m^2n-7mn^2 \\
\qquad -5m^2n-6mn^2+7n^3 \\
\hline
5m^3+m^2n-13mn^2+7n^3
\end{array}$$

11.
$$\begin{array}{l}
13r^2-17r-45 \\
-r-3 \\
\hline
-13r^3+17r^2+45r \\
\qquad -39r^2+51r+135 \\
\hline
-13r^3-22r^2+96r+135
\end{array}$$

12.
$$\begin{array}{l} 7x^3-8x^2z-9z^2 \\ -x-z \\ \hline -7x^4+8x^3z+9xz^2 \\ \qquad -7x^3z+8x^2z^2+9z^3 \\ \hline -7x^4+x^3z+8x^2z^2+9xz^2+9z^3 \end{array}$$

13.
$$\begin{array}{l} -x^5+x^4y-x^3y^2 \\ -x-y \\ \hline x^6-x^5y+x^4y^2 \\ \quad +x^5y-x^4y^2+x^3y^3 \\ \hline x^6 \qquad\qquad +x^3y^3 \end{array}$$

14.
$$\begin{array}{l} -x^3-x^2y-xy^2-y^3 \\ -x-y \\ \hline x^4+x^3y+x^2y^2+xy^3 \\ \quad +x^3y+x^2y^2+xy^3+y^4 \\ \hline x^4+2x^3y+2x^2y^2+2xy^3+y^4 \end{array}$$

X.

12. $(x^2+2x-3)^2 = x^4+4x^2+9+4x^3-6x^2-12x = x^4+4x^3-2x^2-12x+9.$

13. $(x^2-6x+7)^2 = x^4+36x^2+49-12x^3+14x^2-84x$
$= x^4-12x^3+50x^2-84x+49.$

14. $(2x^2-7x+9)^2 = 4x^4+49x^2+81-28x^3+36x^2-126x$
$= 4x^4-28x^3+85x^2-126x+81.$

16. $(x^4-4x^2y^2+y^4)^2 = x^8+16x^4y^4+y^8-8x^6y^2+2x^4y^4-8x^2y^6$
$= x^8-8x^6y^2+18x^4y^4-8x^2y^6+y^8.$

27. $(a+b+c)^3 = (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$
$= a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$

28. $(a-b-c)^3 = (a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3$
$= a^3-3a^2b+3ab^2-b^3-3a^2c+6abc-3b^2c+3ac^2-3bc^2-c^3.$

29. $(m+n)(m-n)(m+n)(m-n) = (m^2-n^2)(m^2-n^2)$
$= m^4-2m^2n^2+n^4.$

30. $(m^2+2mn+n^2)(m^2-n^2) = m^4+2m^3n+m^2n^2-m^2n^2-2mn^3-n^4$
$= m^4+2m^3n-2mn^3-n^4.$

XIV.

1. $$\begin{array}{l} x+10)x^2+15x+50(x+5 \\ \underline{x^2+10x} \\ \qquad 5x+50 \\ \qquad \underline{5x+50} \end{array}$$

2. $$\begin{array}{l} x-7)x^2-17x+70(x-10 \\ \underline{x^2-7x} \\ \qquad -10x+70 \\ \qquad \underline{-10x+70} \end{array}$$

3. $$\begin{array}{l} x-3)x^2+x-12(x+4 \\ \underline{x^2-3x} \\ \qquad 4x-12 \\ \qquad \underline{4x-12} \end{array}$$

4. $$\begin{array}{l} x+1)x^2+13x+12(x+12 \\ \underline{x^2+x} \\ \qquad 12x+12 \\ \qquad \underline{12x+12} \end{array}$$

5. $$\begin{array}{l} x+6)x^3+13x^2+54x+72(x^2+7x+12 \\ \underline{x^3+6x^2} \\ \qquad 7x^2+54x \\ \qquad \underline{7x^2+42x} \\ \qquad\qquad 12x+72 \\ \qquad\qquad \underline{12x+72} \end{array}$$

6. $$\begin{array}{l} x+1)x^3+x^2-x-1(x^2-1 \\ \underline{x^3+x^2} \\ \qquad -x-1 \\ \qquad \underline{-x-1} \end{array}$$

7. $$\begin{array}{l} x+1)x^3+2x^2+2x+1(x^2+x+1 \\ \underline{x^3+x^2} \\ \qquad x^2+2x \\ \qquad \underline{x^2+x} \\ \qquad\qquad x+1 \\ \qquad\qquad \underline{x+1} \end{array}$$

8. $$\begin{array}{l} x^2+3x+1)x^5-5x^3+7x^2+6x+1(x^3-3x^2+3x+1 \\ \underline{x^5+3x^4+x^3} \\ \qquad -3x^4-6x^3+7x^2 \\ \qquad \underline{-3x^4-9x^3-3x^2} \\ \qquad\qquad 3x^3+10x^2+6x \\ \qquad\qquad \underline{3x^3+9x^2+3x} \\ \qquad\qquad\qquad x^2+3x+1 \\ \qquad\qquad\qquad \underline{x^2+3x+1} \end{array}$$

9. $$\begin{array}{l} x^2-2x-1)x^4-4x^3+2x^2+4x+1(x^2-2x-1 \\ \underline{x^4-2x^3-x^2} \\ \quad -2x^3+3x^2+4x \\ \quad \underline{-2x^3+4x^2+2x} \\ \qquad\quad -x^2+2x+1 \\ \qquad\quad \underline{-x^2+2x+1} \end{array}$$

10. $$\begin{array}{l} x^2-2x+1)x^4-4x^3+6x^2-4x+1(x^2-2x+1 \\ \underline{x^4-2x^3+x^2} \\ \quad -2x^3+5x^2-4x \\ \quad \underline{-2x^3+4x^2-2x} \\ \qquad\qquad x^2-2x-1 \\ \qquad\qquad \underline{x^2-2x+1} \end{array}$$

11. $$\begin{array}{l} x^2+x-1)x^4-x^2+2x-1(x^2-x+1 \\ \underline{x^4+x^3-x^2} \\ \quad -x^3+2x-1 \\ \quad \underline{-x^3-x^2+x} \\ \qquad\quad x^2+x-1 \\ \qquad\quad \underline{x^2+x-1} \end{array}$$

12. $$\begin{array}{l} x+2)x^4-4x^2+8x+16(x^3-2x^2+8 \\ \underline{x^4+2x^3} \\ \quad -2x^3-4x^2 \\ \quad \underline{-2x^3-4x^2} \\ \qquad\qquad 8x+16 \\ \qquad\qquad \underline{8x+16} \end{array}$$

13. $$x+4y)x^3+4x^2y+3xy^2+12y^3(x^2+3y^2$$
$$x^3+4x^2y$$
$$3xy^2+12y^3$$
$$3xy^2+12y^3$$

14. $$a+b)a^4+4a^3b+6a^2b^2+4ab^3+b^4(a^3+3a^2b+3ab^2+b^3$$
$$a^4+a^3b$$
$$3a^3b+6a^2b^2$$
$$3a^3b+3a^2b^2$$
$$3a^2b^2+4ab^3$$
$$3a^2b^2+3ab^3$$
$$ab^3+b^4$$
$$ab^3+b^4$$

15. $$a-b)a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5(a^4-4a^3b+6a^2b^2-4ab^3\div b^4$$
$$a^5-a^4b$$
$$-4a^4b+10a^3b^2$$
$$-4a^4b+4a^3b^2$$
$$6a^3b^2-10a^2b^3$$
$$6a^3b^2-6a^2b^3$$
$$-4a^2b^3+5ab^4$$
$$-4a^2b^3+4ab^4$$
$$ab^4-b^5$$
$$ab^4-b^5$$

16. $$x^2-6x+9)x^4-12x^3+50x^2-84x+45(x^2-6x+5$$
$$x^4-6x^3+9x^2$$
$$-6x^3+41x^2-84x$$
$$-6x^3+36x^2-54x$$
$$5x^2-30x+45$$
$$5x^2-30x+45$$

17. $a^2 - 2ab - 3b^2)\,a^5 - 4a^4b + 4a^3b^2 + 4a^2b^3 - 17ab^4 - 12b^5\,(a^3 - 2a^2b + 3ab^2 + 4b^3$

$$\begin{array}{l}
\underline{a^5 - 2a^4b - 3a^3b^2} \\
-2a^4b + 7a^3b^2 + 4a^2b^3 \\
\underline{-2a^4b + 4a^3b^2 + 6a^2b^3} \\
3a^3b^2 - 2a^2b^3 - 17ab^4 \\
\underline{3a^3b^2 - 6a^2b^3 - 9ab^4} \\
4a^2b^3 - 8ab^4 - 12b^5 \\
\underline{4a^2b^3 - 8ab^4 - 12b^5}
\end{array}$$

18. $2ax^2 - 3a^2x + a^3)\,4a^2x^4 - 12a^3x^3 + 13a^4x^2 - 6a^5x + a^6\,(2ax^2 - 3a^2x + a^3$

$$\begin{array}{l}
\underline{4a^2x^4 - 6a^3x^3 + 2a^4x^2} \\
-6a^3x^3 + 11a^4x^2 - 6a^5x \\
\underline{-6a^3x^3 + 9a^4x^2 - 3a^5x} \\
2a^4x^2 - 3a^5x + a^6 \\
\underline{2a^4x^2 - 3a^5x + a^6}
\end{array}$$

19. $x^2 + x - 1)\,x^4 - x^2 + 2x - 1\,(x^2 - x + 1$

$$\begin{array}{l}
\underline{x^4 + x^3 - x^2} \\
-x^3 + 2x - 1 \\
\underline{-x^3 - x^2 + x} \\
x^2 + x - 1 \\
\underline{x^2 + x - 1}
\end{array}$$

20. $x^2 + 2a^2)\,x^4 + a^2x^2 - 2a^4\,(x^2 - a^2$

$$\begin{array}{l}
\underline{x^4 + 2a^2x^2} \\
-a^2x^2 - 2a^4 \\
\underline{-a^2x^2 - 2a^4}
\end{array}$$

21. $x - 15y)\,x^2 - 13xy - 30y^2\,(x + 2y$

$$\begin{array}{l}
\underline{x^2 - 15xy} \\
2xy - 30y^2 \\
\underline{2xy - 30y^2}
\end{array}$$

22. $x + y)\,x^5 + y^5\,(x^4 - x^3y + x^2y^2 - xy^3 + y^4$

$$\begin{array}{l}
\underline{x^5 + x^4y} \\
-x^4y + y^5 \\
\underline{-x^2y - x^3y^2} \\
x^3y^2 + y^5 \\
\underline{x^3y^2 + x^2y^3} \\
-x^2y^3 + y^5 \\
\underline{-x^2y^3 - xy^4} \\
xy^4 + y^5 \\
\underline{xy^4 + y^5}
\end{array}$$

23.
$$
\begin{array}{l}
x-y)\,x^6-y^6\,(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5 \\
\underline{x^6-x^5y} \\
\quad x^5y-y^6 \\
\quad \underline{x^5y-x^4y^2} \\
\qquad x^4y^2-y^6 \\
\qquad \underline{x^4y^2-x^3y^3} \\
\qquad\quad x^3y^3-y^6 \\
\qquad\quad \underline{x^3y^3-x^2y^4} \\
\qquad\qquad x^2y^4-y^6 \\
\qquad\qquad \underline{x^2y^4-xy^5} \\
\qquad\qquad\quad xy^5-y^6 \\
\qquad\qquad\quad \underline{xy^5-y^6}
\end{array}
$$

24.
$$
\begin{array}{l}
a-b+c)\,a^2-b^2+2bc-c^2\,(a+b-c \\
\underline{a^2-ab+ac} \\
\quad ab-b^2-ac+2bc-c^2 \\
\quad \underline{ab-b^2+bc} \\
\qquad -ac+bc-c^2 \\
\qquad \underline{-ac+bc-c^2}
\end{array}
$$

25.
$$
\begin{array}{l}
-1+b)\,b-3b^2+3b^3-b^4\,(-b+2b^2-b^3 \\
\underline{b-b^2} \\
\quad -2b^2+3b^3 \\
\quad \underline{-2b^2+2b^3} \\
\qquad b^3-b^4 \\
\qquad \underline{b^3-b^4}
\end{array}
$$

26.
$$
\begin{array}{l}
a+b-c-d)\,a^2-2ad+2bc-b^2-c^2+d^2\,(a-b+c-d \\
\underline{a^2+ab-ac-ad} \\
\quad -ab-b^2+ac-ad+2bc-c^2+d^2 \\
\quad \underline{-ab-b^2+bc+bd} \\
\qquad ac+bc-bd-ad-c^2+d^2 \\
\qquad \underline{ac+bc-c^2-cd} \\
\qquad\quad -ad-bd+cd+d^2 \\
\qquad\quad \underline{ad-bd+cd+d^2}
\end{array}
$$

27. $x+y+z)\,x^3+y^3+z^3-3xyz\,(x^2-xy-xz+y^2-yz+z^2$

$x^3+x^2y+x^2z$

$-x^2y-x^2z-3xyz+y^3+z^3$

$-x^2y-xy^2-xyz$

$-x^2z+xy^2-2xyz+y^3+z^3$

$-x^2z-xyz-xz^2$

$xy^2-xyz+xz^2+y^3+z^3$

$xy^2+y^3+y^2z$

$-xyz-y^2z+xz^2+z^3$

$-xyz-y^2z-yz^2$

$xz^2+yz^2+z^3$

$xz^2+yz^2+z^3$

28. $x^3+y^2)\,x^{15}+y^{10}\,(x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8$

$x^{15}+x^{12}y^2$

$-x^{12}y^2+y^{10}$

$-x^{12}y^2-x^9y^4$

$x^9y^4+y^{10}$

$x^9y^4+x^6y^6$

$-x^6y^6+y^{10}$

$-x^6y^6-x^3y^8$

$x^3y^8+y^{10}$

$x^3y^8+y^{10}$

29. $p-q+3r)\,p^2+pq+2pr-2q^2+7qr-3r^2\,(p+2q-r$

$p^2-pq+3pr$

$2pq-pr-2q^2+7qr$

$2pq-2q^2+6qr$

$-pr+qr-3r^2$

$-pr+qr-3r^2$

30. $a^4+a^3b+a^2b^2+ab^3+b^4)\,a^8+a^6b^2+a^4b^4+a^2b^6+b^8\,(a^4-a^3b+a^2b^2-ab^3+b^4$

$$\underline{a^8+a^7b\;+a^6b^2+a^5b^3+a^4b^4}$$
$$-a^7b\;-a^5b^3+a^2b^6+b^8$$
$$\underline{-a^7b\;-a^6b^2-a^5b^3-a^4b^4-a^3b^5}$$
$$a^6b^2+a^4b^4+a^3b^5+a^2b^6+b^8$$
$$\underline{a^6b^2+a^5b^3+a^4b^4+a^3b^5+a^2b^6}$$
$$-a^5b^3+b^8$$
$$\underline{-a^5b^3-a^4b^4-a^3b^5-a^2b^6-ab^7}$$
$$a^4b^4+a^3b^5+a^2b^6+ab^7+b^8$$
$$\underline{a^4b^4+a^3b^5+a^2b^6+ab^7+b^8}$$

31. $x^4-x^3y+x^2y^2-xy^3+y^4)\,x^8+x^6y^2+x^4y^4+x^2y^6+y^8\,(x^4+x^3y+x^2y^2+xy^3+y^4$

$$\underline{x^8-x^7y\;+x^6y^2-x^5y^3+x^4y^4}$$
$$x^7y\;+x^5y^3+x^2y^6+y^8$$
$$\underline{x^7y\;-x^6y^2+x^5y^3-x^4y^4+x^3y^5}$$
$$x^6y^2+x^4y^4-x^3y^5+x^2y^6+y^8$$
$$\underline{x^6y^2-x^5y^3+x^4y^4-x^3y^5+x^2y^6}$$
$$x^5y^3+y^8$$
$$\underline{x^5y^3-x^4y^4+x^3y^5-x^2y^6+xy^7}$$
$$x^4y^4-x^3y^5+x^2y^6-xy^7+y^8$$
$$\underline{x^4y^4-x^3y^5+x^2y^6-xy^7+y^8}$$

32. $2x^2+3x+2)\,4x^5-x^3+4x\,(2x^3-3x^2+2x$

$$\underline{4x^5+6x^4+4x^3}$$
$$-6x^4-5x^3+4x$$
$$\underline{-6x^4-9x^3-6x^2}$$
$$4x^3+6x^2+4x$$
$$\underline{4x^3+6x^2+4x}$$

33. $a-3)\,a^5-243\,(a^4+3a^3+9a^2+27a+81$

$$\begin{array}{l}
a^5-3a^4 \\ \hline
\quad 3a^4-243 \\
\quad 3a^4-9a^3 \\ \hline
\qquad 9a^3-243 \\
\qquad 9a^3-27a^2 \\ \hline
\qquad\quad 27a^2-243 \\
\qquad\quad 27a^2-81a \\ \hline
\qquad\qquad 81a-243 \\
\qquad\qquad 81a-243 \\ \hline
\end{array}$$

34. $k^3-1)\,k^{10}-k\,(k^7+k^4+k$

$$\begin{array}{l}
k^{10}-k^7 \\ \hline
\quad k^7-k \\
\quad k^7-k^4 \\ \hline
\qquad k^4-k \\
\qquad k^4-k \\ \hline
\end{array}$$

35. $x+4)\,x^3-5x^2-46x-40\,(x^2-9x-10$

$$\begin{array}{l}
x^3+4x^2 \\ \hline
\quad -9x^2-46x \\
\quad -9x^2-36x \\ \hline
\qquad -10x-40 \\
\qquad -10x-40 \\ \hline
\end{array}$$

36. $2x-3a)\,48x^3-76ax^2-64a^2x+105a^3\,(24x^2-2ax-35a^2$

$$\begin{array}{l}
48x^3-72ax^2 \\ \hline
\quad -4ax^2-64a^2x \\
\quad -4ax^2+6a^2x \\ \hline
\qquad -70a^2x+105a^3 \\
\qquad -70a^2x+105a^3 \\ \hline
\end{array}$$

37. $3x^2-4x+5)\,18x^4-45x^3+82x^2-67x+40\,(6x^2-7x+8$

$$\begin{array}{l}
18x^4-24x^3+30x^2 \\ \hline
\quad -21x^3+52x^2-67x \\
\quad -21x^3+28x^2-35x \\ \hline
\qquad 24x^2-32x+40 \\
\qquad 24x^2-32x+40 \\ \hline
\end{array}$$

38. $2x-3a)16x^4-72a^2x^2+81a^4(8x^3+12ax^2-18a^2x-27a^3$

$16x^4-24ax^3$

$24ax^3-72a^2x^2$

$24ax^3-36a^2x^2$

$-36a^2x^2+81a^4$

$-36a^2x^2+54a^3x$

$-54a^3x+81a^4$

$-54a^3x+81a^4$

39. $3x+4a)81x^4-256a^4(27x^3-36ax^2+48a^2x-64a^3$

$81x^4+108ax^3$

$-108ax^3-256a^4$

$-108ax^3-144a^2x^2$

$144a^2x^2-256a^4$

$144a^2x^2+192a^3x$

$-192a^3x-256a^4$

$-192a^3x-256a^4$

40. $a^2-b^2)2a^3+3a^2b-2ab^2-3b^3(2a+3b$

$2a^3-2ab^2$

$3a^2b-3b^3$

$3a^2b-3b^3$

41. $x^2-a^2)x^3+2ax^2-a^2x-2a^3(x+2a$

x^3-a^2x

$2ax^2-2a^3$

$2ax^2-2a^3$

42. $a^2+3b^2)a^4-a^2b^2-12b^4(a^2-4b^2$

$a^4+3a^2b^2$

$-4a^2b^2-12b^4$

$-4a^2b^2-12b^4$

43. $x^2+3x+y)x^4-9x^2-6xy-y^2(x^2-3x-y$

$$\underline{x^4+3x^3+x^2y}$$
$$-3x^3-x^2y-9x^2-6xy-y^2$$
$$\underline{-3x^3-9x^2-3xy}$$
$$-x^2y-3xy-y^2$$
$$\underline{-x^2y-3xy-y^2}$$

44. $x^2-3xy+2y^2)x^4-6x^3y+9x^2y^2-4y^4(x^2-3xy-2y^2$

$$\underline{x^4-3x^3y+2x^2y^2}$$
$$-3x^3y+7x^2y^2-4y^4$$
$$\underline{-3x^3y+9x^2y^2-6xy^3}$$
$$-2x^2y^2+6xy^3-4y^4$$
$$\underline{-2x^2y^2+6xy^3-4y^4}$$

45. $x-3y)x^4-81y^4(x^3+3x^2y+9xy^2+27y^3$

$$\underline{x^4-3x^3y}$$
$$3x^3y-81y^4$$
$$\underline{3x^3y-9x^2y^2}$$
$$9x^2y^2-81y^4$$
$$\underline{9x^2y^2-27xy^3}$$
$$27xy^3-81y^4$$
$$\underline{27xy^3-81y^4}$$

46. $a-2b)a^4-16b^4(a^3+2a^2b+4ab^2+8b^3$

$$\underline{a^4-2a^3b}$$
$$2a^3b-16b^4$$
$$\underline{2a^3b-4a^2b^2}$$
$$4a^2b^2-16b^4$$
$$\underline{4a^2b^2-8ab^3}$$
$$8ab^3-16b^4$$
$$\underline{8ab^3-16b^4}$$

47. $3a+2b)81a^4-16b^4(27a^3-18a^2b+12ab^2-8b^3$

$$\begin{array}{l}
\underline{81a^4+54a^3b} \\
\quad -54a^3b-16b^4 \\
\quad \underline{-54a^3b-36a^2b^2} \\
\qquad\quad 36a^2b^2-16b^4 \\
\qquad\quad \underline{36a^2b^2+24ab^3} \\
\qquad\qquad\quad -24ab^3-16b^4 \\
\qquad\qquad\quad \underline{-24ab^3-16b^4}
\end{array}$$

48. $2x+3y)16x^4-81y^4(8x^3-12x^2y+18xy^2-27y^3$

$$\begin{array}{l}
\underline{16x^4+24x^3y} \\
\quad -24x^3y-81y^4 \\
\quad \underline{-24x^3y-36x^2y^2} \\
\qquad\quad 36x^2y^2-81y^4 \\
\qquad\quad \underline{36x^2y^2+54xy^3} \\
\qquad\qquad\quad -54xy^3-81y^4 \\
\qquad\qquad\quad \underline{-54xy^3-81y^4}
\end{array}$$

49. $a+2b+3c)3a^2+8ab+4b^2+10ac+8bc+3c^2(3a+2b+c$

$$\begin{array}{l}
\underline{3a^2+6ab+9ac} \\
\quad 2ab+4b^2+ac+8bc \\
\quad \underline{2ab+4b^2+6bc} \\
\qquad\quad ac+2bc+3c^2 \\
\qquad\quad \underline{ac+2bc+3c^2}
\end{array}$$

50. $a^2+2ax+4x^2)a^4+4a^2x^2+16x^4(a^2-2ax+4x^2$

$$\begin{array}{l}
\underline{a^4+2a^3x+4a^2x^2} \\
\quad -2a^3x+16x^4 \\
\quad \underline{-2a^3x-4a^2x^2-8ax^3} \\
\qquad\quad 4a^2x^2+8ax^3+16x^4 \\
\qquad\quad \underline{4a^2x^2+8ax^3+16x^4}
\end{array}$$

51. $$\begin{array}{l} x^2 - xy + y^2)\, x^4 + x^2y^2 + y^4\, (x^2 + xy + y^2 \\ \qquad \underline{x^4 - x^3y + x^2y^2} \\ \qquad\quad x^3y + y^4 \\ \qquad\quad \underline{x^3y - x^2y^2 + xy^3} \\ \qquad\qquad x^2y^2 - xy^3 + y^4 \\ \qquad\qquad \underline{x^2y^2 - xy^3 + y^4} \end{array}$$

52. $$\begin{array}{l} 16x^2 + 4xy + y^2)\, 256x^4 + 16x^2y^2 + y^4\, (16x^2 - 4xy + y^2 \\ \qquad \underline{256x^4 + 64x^3y + 16x^2y^2} \\ \qquad\quad - 64x^3y + y^4 \\ \qquad\quad \underline{- 64x^3y - 16x^2y^2 - 4xy^3} \\ \qquad\qquad 16x^2y^2 + 4xy^3 + y^4 \\ \qquad\qquad \underline{16x^2y^2 + 4xy^3 + y^4} \end{array}$$

53. $$\begin{array}{l} x^3 + x - y)\, x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3\, (x^2 + xy - y^2 \\ \qquad \underline{x^5 + x^3 - x^2y} \\ \qquad\quad x^4y - x^3y^2 + x^2y - 2xy^2 + y^3 \\ \qquad\quad \underline{x^4y + x^2y - xy^2} \\ \qquad\quad - x^3y^2 - xy^2 + y^3 \\ \qquad\quad \underline{- x^3y^2 - xy^2 + y^3} \end{array}$$

54. $$\begin{array}{l} x - a)\, ax^3 + 3a^2x^2 - 2a^3x - 2a^4\, (ax^2 + 4a^2x + 2a^3 \\ \qquad \underline{ax^3 - a^2x^2} \\ \qquad\quad 4a^2x^2 - 2a^3x \\ \qquad\quad \underline{4a^2x^2 - 4a^3x} \\ \qquad\qquad 2a^3x - 2a^4 \\ \qquad\qquad \underline{2a^3x - 2a^4} \end{array}$$

55. $$\begin{array}{l} a + x)\, a^2 - x^2\, (a - x \\ \qquad \underline{a^2 + ax} \\ \qquad\quad - ax - x^2 \\ \qquad\quad \underline{- ax - x^2} \end{array}$$

56. $$\begin{array}{l} 2x + 3y + z)\, 2x^2 + xy - 3y^2 - 4yz - xz - z^2\, (x - y - z \\ \qquad \underline{2x^2 + 3xy + xz} \\ \qquad\quad - 2xy - 3y^2 - 4yz - 2xz - z^2 \\ \qquad\quad \underline{- 2xy - 3y^2 - yz} \\ \qquad\qquad - 2xz - 3yz - z^2 \\ \qquad\qquad \underline{- 2xz - 3yz - z^2} \end{array}$$

57. $1+5x+x^2)\,2+9x+14x^3+3x^4\,(2-x+3x^2$

$$\begin{array}{l}
2+10x+2x^2 \\ \hline
-x-2x^2+14x^3 \\
-x-5x^2-x^3 \\ \hline
3x^2+15x^3+3x^4 \\
3x^2+15x^3+3x^4 \\ \hline
\end{array}$$

58. $3-5x+7x^2)\,12-38x+82x^2-112x^3+106x^4-70x^5\,(4-6x+8x^2-10x^3$

$$\begin{array}{l}
12-20x+28x^2 \\ \hline
-18x+54x^2-112x^3 \\
-18x+30x^2-42x^3 \\ \hline
24x^2-70x^3+106x^4 \\
24x^2-40x^3+56x^4 \\ \hline
-30x^3+50x^4-70x^5 \\
-30x^3+50x^4-70x^5 \\ \hline
\end{array}$$

59. $x^4-x^3y+x^2y^2-xy^3+y^4)\,x^5+y^5\,(x+y$

$$\begin{array}{l}
x^5-x^4y+x^3y^2-x^2y^3+xy^4 \\ \hline
x^4y-x^3y^2+x^2y^3-xy^4+y^5 \\
x^4y-x^3y^2+x^2y^3-xy^4+y^5 \\ \hline
\end{array}$$

60. $ax+xy+ab+by)\,a^2x^2-a^2b^2-x^2y^2+b^2y^2\,(ax-xy-ab+by$

$$\begin{array}{l}
a^2x^2+ax^2y+a^2bx+abxy \\ \hline
-ax^2y-a^2bx-abxy-a^2b^2-x^2y^2+b^2y^2 \\
-ax^2y-x^2y^2-abxy-bxy^2 \\ \hline
-a^2bx+bxy^2-a^2b^2+b^2y^2 \\
-a^2bx-abxy-a^2b^2-ab^2y \\ \hline
abxy+bxy^2+ab^2y+b^2y^2 \\
abxy+bxy^2+ab^2y+b^2y^2 \\ \hline
\end{array}$$

61. $ax+by)\,abx^2+a^2xy+b^2xy+aby^2\,(bx+ay$

$$abx^2+b^2xy$$
$$a^2xy+aby^2$$
$$a^2xy+aby^2$$

62. $x^2+ax+b^2)\,x^4+2b^2x^2-a^2x^2+b^4\,(x^2-ax+b^2$

$$x^4+ax^3+b^2x^2$$
$$-ax^3+b^2x^2-a^2x^2$$
$$-ax^3-a^2x^2-ab^2x$$
$$b^2x^2+ab^2x+b^4$$
$$b^2x^2+ab^2x+b^4$$

XV.

1. $x^2-ax+c)\,x^4-(a^2-b-c)x^2-(b-c)ax+bc\,(x^2+ax+b$

$$x^4-ax^3+cx^2$$
$$ax^3-(a^2-b)x^2-(ab-ac)x+bc$$
$$ax^3-a^2x^2+acx$$
$$bx^2-abx+bc$$
$$bx^2-abx+bc$$

2. $y-n)\,y^3-(l+m+n)y^2+(lm+ln+mn)y-lmn\,(y^2-(l+m)y+lm$

$$y^3-ny^2$$
$$-(l+m)y^2+(lm+ln+mn)y-lmn$$
$$-(l+m)y^2+(ln+mn)y$$
$$lmy-lmn$$
$$lmy-lmn$$

3. $x^3 - mx^2 + nx + r)x^5 - (m-c)x^4 + (n - cm + d)x^3 + (r + cn - dm)x^2 +$

$$(cr + dn)x + dr(x^2 + cx + d$$

$$x^5 - mx^4 + nx^3 + rx^2$$

$$cx^4 + (d - cm)x^3 + (cn - dm)x^2$$

$$cx^4 - cmx^3 + cnx^2 + crx$$

$$dx^3 - dmx^2 + dnx + dr$$

$$dx^3 - dmx^2 + dnx + dr$$

4. $x^2 + 5x - 4)x^4 + (5 + a)x^3 - (4 - 5a + b)x^2 - (4a + 5b)x + 4b(x^2 + ax - b$

$$x^4 + 5x^3 - 4x^2$$

$$ax^3 + (5a - b)x^2 - (4a + 5b)x$$

$$ax^3 + 5ax^2 - 4ax$$

$$-bx^2 - 5bx + 4b$$

$$-bx^2 - 5bx + 4b$$

5. $x^2 - (a + c)x + ac)x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2$

$$-(abc + abd + acd + bcd)x + abcd(x^2 - (b + d)x + bd$$

$$x^4 - (a + c)x^3 + acx^2$$

$$-(b + d)x^3 + (ab + ad + bc + bd + cd)x^2$$

$$-(b + d)x^3 + (ab + ad + bc + cd)x^2 - (abc + acd)x$$

$$bdx^2 - (abd + bcd)x + abcd$$

$$bdx^2 - (abd + bcd)x + abcd$$

XVIII.

1. $(x^2 - ax) - (bx - ab) = x(x - a) - b(x - a) = (x - b)\ (x - a).$

2. $(ab + ax) - (bx + x^2) = a(b + x) - x(b + x) = (a - x)\ (b + x).$

3. $(bc + by) - (cy + y^2) = b(c + y) - y(c + y) = (b - y)\ (c + y).$

4. $(bm + mn) + (ab + an) = m(b + n) + a(b + n) = (a + m)\ (b + n).$

5. $(abx^2-axy)+(bxy-y^2)=ax(bx-y)+y(bx-y)=(ax+y)(bx-y)$.

6. $(abx-aby)+(cdx-cdy)=ab(x-y)+cd(x-y)=(ab+cd)(x-y)$.

7. $(cdx^2+dmxy)-(cnxy+mny^2)=dx(cx+my)-ny(cx+my)$
$=(cx+my)(dx-ny)$.

8. $(abcx-b^2dx)-(acdy-bd^2y)=bx(ac-bd)-dy(ac-bd)$
$=(ac-bd)(bx-dy)$.

XXVI.

15. $(x^2-2xy+y^2)-z^2=(x-y)^2-z^2=(x-y+z)(x-y-z)$.

17. $(a^2-2ac+c^2)-(b^2+2bd+d^2)=(a-c)^2-(b+d)^2$, etc.

18. $a^2-(b^2-2bc+c^2)=a^2-(b-c)^2$, etc.

19. $(x^2+2xy+y^2)-z^2=(x+y)^2-z^2$, etc.

20. $(a^2-2ab+b^2)-(m^2-2mn+n^2)=(a-b)^2-(m-n)^2$, etc.

23. $1-(a^2-2ab+b^2)=1-(a-b)^2$, etc.

24. $1-(x^2-2xy+y^2)=1-(x-y)^2$, etc.

25. $x^2-(y^2+2yz+z^2)=x^2-(y+z)^2$, etc.

26. $a^2-(4b^2-12bc+9c^2)=a^2-(2b-3c)^2$, etc.

29. $(a^2-2ab+b^2)-(c^2+2cd+d^2)=(a-b)^2-(c+d)^2$, etc.

30. $(a^2-2ac+c^2)-(b^2-2bd+d^2)=(a-c)^2-(b-d)^2$, etc.

31. $3ax(a^2x^2-9)=3ax(ax+3)(ax-3)$.

XXVII.

11. $x^6-y^6=(x^3+y^3)(x^3-y^3)=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$.

12. $x^6-1=(x^3+1)(x^3-1)=(x+1)(x^2-x+1)(x-1)(x^2+x+1)$.

13. $a^6 - 64 = a^6 - 2^6 = (a^3 + 2^3)(a^3 - 2^3)$
$= (a+2)(a^2 - 2a + 4)(a-2)(a^2 + 2a + 4)$.

14. $729 - y^6 = 3^6 - y^6 = (3^3 + y^3)(3^3 - y^3)$
$= (3+y)(9 - 3y + y^2)(3-y)(9 + 3y + y^2)$.

XXX.

1. $6 + 4 - 5 - 3 = 2$.

2. $6 + 3 - 5 - 4 = 0$.

3. $18 + 12 - 5 - 8 = 17$.

4. $3 \times 11 - 2 \times 1 = 33 - 2 = 31$.

5. $10 \times 2 = 20$.

6. $12 + 3 \times 7 = 12 + 21 = 33$.

7. $(12+3)(4+3) = 15 \times 7 = 105$.

8. $12 + 12 + 3 = 27$.

9. $(25+3) \div (6-4) = 28 \div 2 = 14$.

10. $6 \times 5 \times 4 = 120$.

11. $6 \times 5 \times 7 = 210$.

12. $6 \times 3 \times 9 \times 9 = 1458$.

13. $6 \times 5 \times 1 \times 1 = 30$.

14. $\sqrt{5 \times 5} = 5$.

15. $\sqrt{y^2} = y = 3$.

16. $(\sqrt{x})^2 = (2)^2 = 4$.

17. $(\sqrt{x} + b)^2 = (2+5)^2 = 49$.

18. $\sqrt{5 \times 5 \times 4} = \sqrt{100} = 10$.

19. $\sqrt{2 \times 6 \times 4 \times 3} = \sqrt{144} = 12.$

20. $(36+25+3) \div (4+9+3) = 64 \div 16 = 4.$

21. $18 + (8-3)^2 = 18 + 25 = 43.$

22. $\{6-(5-3)\}\{6-(4-3)\} = 4 \times 5 = 20.$

23. $(6-5-3)^2 + (6-4+3)^2 = 4 + 25 = 29.$

24. $3 \times 8^3 + 4 \times 10^4 = 1536 + 40000$, etc.

25. $3 \times 1^2 + 7^2 = 3 + 49 = 52.$

XXXI.

1. $3 \times 3 \times 2 \times 1 - 3^3 + 2^3 + 1^3 = 18 - 27 + 8 + 1 = 0.$

2. $3^3 + 2^3 - 5^3 + 3 \times 3 \times 2 \times 5 = 27 + 8 - 125 + 90 = 0.$

3. $(a^2 + 2ac + c^2) - (a^2 + c^2) = 2ac.$

4. $x^2 + y^2 - (x^2 - 2xy + y^2) = 2xy.$

5. $(a^2 + 2ab + b^2)x - (a^2 + 2abx + b^2x^2) = (a^2 + b^2)x - a^2 - b^2x^2.$

6. Multiply $2x - m$ by $x + 2m$; the result is $2x^2 + 3mx - 2m^2$.
Multiply $2x + n$ by $x - 2n$; the result is $2x^2 - 3nx - 2n^2$
Then multiply the two results together.

7.
$$
\begin{array}{l}
ar + b)\,acr^3 + (bc+ad)r^2 + (bd+ae)r + be\,(cr^2 + dr + e \\
\qquad \underline{acr^3 + bcr^2} \\
\qquad\qquad adr^2 + (bd+ae)r \\
\qquad\qquad \underline{adr^2 + bdr} \\
\qquad\qquad\qquad aer + be \\
\qquad\qquad\qquad \underline{aer + be}
\end{array}
$$

The divisor $= 10 + 1$, or 11; the dividend $= 1000 + 200 + 20 + 1$, or 1221; the quotient $= 100 + 10 + 1$, or 111; and $1221 \div 11 = 111$.

8. Multiply $a+b+c$ by $a+b-c$; the result is $(a+b)^2-c^2$, or $a^2+2ab+b^2-c^2$.

Multiply $c+b-a$ by $c+a-b$; the result is $c^2-(a-b)^2$, or $c^2-a^2+2ab-b^2$.

Then $(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)=(2ab)^2-(a^2+b^2-c^2)^2$
$=4a^2b^2-a^4-b^4-c^4-2a^2b^2+2a^2c^2+2b^2c^2$
$=-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2$.

9. $(a+d)(d+c)-(c+d)(d+a)-(a+c)(d-d)$
$=ad+ac+d^2+cd-cd-ac-d^2-ad-0=0$.

10. $0+(4-16)+\{16-(0+6)\}+\{12-(0+6)\}^2$
$=-12+10+6^2=-2+36=34$.

11. $\{4-3+2-1\}\{4+3-2-1\}=2\times4=8$;
and $16-9-4+1+2\times2=8$.

18. $(ac-ad-bc+bd)-(bc-bd-ac+ad)=2ac-2ad-2bc+2bd$.

19. $ab+ay+bx+xy+x-y+ax-by+ax+ay=$ etc.

20. $(x^2+3x+2)-(x^2-3x+2)=6x$, etc.

21. $ax-by+x-y+x^2-xy+ab-ay-bx+xy$, etc.

22. $(6x^2+20x+16)-(6x^2-28x+16)=48x$, etc.

23. $2mx-3ny+x+y+4mx+4nx-4my-4ny+mx+ny=$ etc.

24. $x^2+y^2+z^2+2xy+2xz+2yz+x^2+y^2+z^2$
$=(x^2+2xy+y^2)+(y^2+2yz+z^2)+(x^2+2xz+z^2)$
$=(x+y)^2+(y+z)^2+(x+z)^2$.

25. $4a^2+6ac-4ab+6ab+9bc-6b^2=4a^2+6ac+2ab+9bc-6b^2$.

26. $\dfrac{ab-cd}{cd+e}=\dfrac{63-15}{15+1}=\dfrac{48}{16}=3$.

$(bc-ad)(bd-ce)=(35-27)(21-5)=8\times16=128$.

$\dfrac{b^2-c^2}{c+d}=\dfrac{49-25}{5+3}=\dfrac{24}{8}=3$.

$d^e-c^d=3^5-5^3=243-125=118$.

27. $0-0+2^3+1^3=8+1=9$.

28. $48+4-8=44$.

29. $(1-2-3)^2+(2-1-3)^2+(3-1-2)^2=(-4)^2+(-2)^2+0=16+4$
$=20$.

30. $(1+2-4)^2+(1-2+4)^2+(2+4-1)^2=(-1)^2+(3)^2+(5)^2$
$=1+9+25=35$.

31. $(-1+2)^2+(2-3)^2+(-1-3)^2=(1)^2+(-1)^2+(-4)^2=1+1+16$
$=18$.

32. Let x and y be the numbers; then $(x^2-y^2)\div(x+y)=x-y$.

33. Let x and y be the numbers; then $(x+y)(x-y)=x^2-y^2$.

34. Let x and $x+1$ be the integers; then $(x+x+1)^2=4x^2+4x+1$
$=4x(x+1)+1$.

35. Let x and $x+2$ be the two even numbers; then $x+1$ is the odd number between them; and $(x+x+2)^2=4x^2+8x+4=4(x+1)^2$.

36. Let x and y be the parts; then $x+y=2$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=2(x-y)$.

37. Let x and y be the parts; then $x+y=50$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=50(x-y)$.

38. Let x and y be the parts; then $x+y=n$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=n(x-y)$.

39. Let x and $x+1$ be the numbers; then $x(x+1)+x^2+(x+1)^2$
$=x^2+x+x^2+x^2+2x+1=3x^2+3x+1=(x+1)^3-x^3$.

40. Let $x-1, x, x+1$ be the three numbers; then $(x-1)^3+x^3+(x+1)^3$
$=x^3-3x^2+3x-1+x^3+x^3+3x^2+3x+1=3x^3+6x=3x(x^2+2)$.

XXXII.

1. $7x-5x=11-5$; $2x=6$; $x=3$.

2. $12x-8x=15-7$; $4x=8$; $x=2$.

3. $236x-97x=564-425$; $139x=139$; $x=1$.

4. $5x-3x=7+7$; $2x=14$; $x=7$.

5. $12x-8x=9-1$; $4x=8$; $x=2$.

6. $124x-112x=43-19$; $12x=24$; $x=2$.

7. $5x-2x=27-18$; $3x=9$; $x=3$.

8. $12x-7x=145-125$; $5x=20$; $x=4$.

9. $14x-8x=80-26$; $6x=54$; $x=9$.

10. $-3x-x=-83-133$; $-4x=-216$; $4x=216$; $x=54$.

11. $-3x-5x=-3-13$; $-8x=-16$; $8x=16$; $x=2$.

12. $9x-12x=100-127$; $-3x=-27$; $3x=27$; $x=9$.

13. $-5x+4x=6-15$; $-x=-9$; $x=9$.

14. $3x-7x=6+22$; $-4x=28$; $4x=-28$; $x=-7$.

15. $4x-12x=-16-8$; $-8x=-24$; $8x=24$; $x=3$.

16. $5x-3x+7=4x-6x+35$; $4x=28$; $x=7$.

17. $6x-18+8x+15x-21=10x-4-16x+35$; $35x=70$; $x=2$.

18. $9x-15x+18+30=0$; $-6x=-48$; $x=8$.

19. $12x-45x-15+42-48x+783=0$; $-81x=-810$; $x=10$.

20. $x-28x+77=14x-70-152+19x-61$; $-60x=-360$; $x=6$.

21. $x^2+4x-21=x^2-20x+75$; $24x=96$; $x=4$

22. $x^2+4x-96=x^2-5x-6$; $9x=90$; $x=10$.

23. $9x-x^2-14+x^2-2x-15-2x+2+12=0$; $5x=15$; $x=3$.

24. $2x^2+3x-35=36-17x+2x^2+229$; $20x=300$; $x=15$.

25. $21-32x+12x^2=12x^2-17x+6$; $-15x=-15$; $x=1$.

26. $14-x-5x^2+5x+30+20-29x+5x^2=45x-76$; $-70x=-140$; $x=2$.

27. $x^2+10x+25-16+8x-x^2=21x$; $-3x=-9$; $x=3$.

28. $5x^2-20x+20+7x^2-42x+63=12x^2-85x+133+42$; $23x=92$; $x=4$.

29. $9x^2-102x+289+16x^2-200x+625-25x^2+290x-841=1$; $-12x=-72$; $x=6$.

30. $x^2-4x-45+x^2+2x-80=2x^2-11x-21-113$; $9x=-9$; $x=-1$.

XXXIII.

1. Let x be the number ; then $2x+14=154$; $x=70$.

2. Let x be the number ; then $4x+16=188$; $x=43$.

3. Let x be the number ; then $x+46=3x$; $x=23$.

4. Let x be the smaller number ; then $3x$ is the greater number ; and $16-x=30-3x$; $2x=14$; $x=7$; $3x=21$.

5. Let x be the first part ; then $x-10$ is the second ; $x-18$ the third ; $x-24$ the fourth ; and $x+x-10+x-18+x-24=92$; $x=36$, etc.

6. Let x be the greater number ; then $20-x$ is the smaller number ; and $3(20-x)+5x=84$; $x=12$, etc.

7. Let x be the father's age in years ; then $80-x$ is the son's age ; and $2(80-x)=x+10$; $x=50$, etc.

8. Let x be the age of the eldest; then $x-20$ is the age of the youngest; and $x=3(x-20)$; $x=30$, etc.

9. Let x be the sum in pounds; then $x+24-80=80-x$; $x=68$.

10. Let x be the price of a yard of cloth in shillings; then $2x$ is the price of a yard of silk; and $30x+80x=66\times 20$; $x=12$, etc.

11. Let x be the number; then $2x+24-80=100-x$; $x=52$.

12. Let x be A's share in pounds; then B's share is $280-x$; C's, $260-x$; D's, $220-x$; and $x+280-x+260-x+220-x=500$; $x=130$, etc.

13. Let x be the number of children; then there are $2x$ women and $4x$ men; and $x+2x+4x=266$; $x=38$, etc.

14. Let x be B's share in pounds; then A's share is $x-100$, and C's is $x+270$; and $x+x-100+x+270=1520$; $x=450$, etc.

15. Let x be the greater; then $x-8$ is the less; and $4(x-8)=2x+10$; $x=21$; $x-8=13$.

16. Let x be what each had at first in pounds; then $3(x+5)=11(x-5)$; $x=$ £8, 15s.

17. Let x be A's age; then $x-58$ is B's age; and $x-60=50-(x-58)$; $x=84$, etc.

18. Let x be A's age; then $x-34$ is B's age; and $x-50=40-(x-34)$; $x=62$, etc.

19. Let x be the share of a daughter; then $2x$ is the share of a son; and $3x+4x+500$ is the share of the wife;
then $3x+4x+3x+4x+500=7500$; $x=500$, etc.

20. Let x be the number of gallons in the vessel at first; then $x+42=7x$; $x=7$; and $42+7=49$, the number of gallons that the vessel held.

21. Let x be the number of pounds A has; then B has $x+10$; C has $x+x+10$; and $x+x+10+x+x+10=76$; $x=14$, etc.

22. Let x be the greater; then $x-14$ is the less; and $x+x-14=48$; $x=31$, etc.

23. Let x be the number of pounds won by A; then $72+x=3(52-x)$; $x=21$.

24. Let x be one of the parts; then $84-x$ is the other; and $3x=4(84-x)$; $x=48$.

25. Let x be one of the parts; then $90-x$ is the other; and $4x=5(90-x)$; $x=50$.

26. Let x be the greater part; then $60-x$ is the less; and $x=60-x+24$; $x=42$.

27. Let x be the greater part; then $84-x$ is the less; and $x-36=84-x$; $x=60$.

28. Let x be one of the parts; then $20-x$ is the other; and $3x+5(20-x)=84$; $x=8$.

29. Let x be B's age; then $2x$ is A's age; and $2x-22=3(x-22)$; $x=44$; $2x=88$.

30. Let x be the number of years; then $30+x=2(6+x)$; $x=18$.

31. Let x be B's age; then $2x$ is A's age; and $2x-20=3(x-20)$; $x=40$.

32. Let x be B's age; then $3x$ is A's age; and $3x+19=2(x+19)$; $x=19$.

33. Let x be the number of years; then $50+x=42+3x$; $x=4$.

34. Let x be the number of guineas; then $x+48$ is the number of half-crowns; and, expressing all the quantities as sixpences, $42x+5(x+48)=100\times 40$; $47x=3760$; $x=80$.

35. Let x be the number of shillings; then $41-x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x+5(41-x)=74\times 2$; $-3x=-57$; $x=19$.

36. Let x be the number of shillings; then $300-x$ is the number of fourpenny pieces; and, expressing all the quantities as fourpenny pieces, $3x+300-x=700$; $x=200$.

37. Let x be the number of moidores; then $x+3$ is the number of sovereigns; and $27x+20(x+3)=50\times 20$; $47x=940$; $x=20$.

38. Let x be the number of shillings; then $6x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x+30x=1696$; $x=53$.

39. Let x be the number of sovereigns; then $2x$ is the number of shillings, and $3x$ the number of pence; and, expressing all the quantities as pence, $240x+24x+3x=1335$; $x=5$.

XXXV.

1. $a^2-b^2=(a-b)(a+b)$; $a^3-b^3=(a-b)(a^2+ab+b^2)$.

2. $a^4-b^4=(a^2-b^2)(a^2+b^2)$.

3. $a^2-x^2=(a+x)(a-x)$; $(a-x)^2=(a-x)(a-x)$.

4. $a^3+x^3=(a+x)(a^2-ax+x^2)$; $(a+x)^3=(a+x)(a+x)(a+x)$.

5. $9x^2-1=(3x+1)(3x-1)$; $(3x+1)^2=(3x+1)(3x+1)$.

6. $1-25a^2=(1+5a)(1-5a)$; $(1-5a)^2=(1-5a)(1-5a)$.

7. $x^2-y^2=(x+y)(x-y)$; $(x+y)^2=(x+y)(x+y)$;
$x^2+3xy+2y^2=(x+y)(x+2y)$.

8. $x^2-y^2=(x+y)(x-y)$; $x^3-y^3=(x-y)(x^2+xy+y^2)$;
$x^2-7xy+6y^2=(x-y)(x-6y)$.

9. $x^2-1=(x-1)(x+1)$; $x^3-1=(x-1)(x^2+x+1)$;
$x^2+x-2=(x-1)(x+2)$.

10. $1-a^2=(1-a)(1+a)$; $1+a^3=(1+a)(1-a+a^2)$;
$a^2+5a+4=(1+a)(4+a)$.

XXXVI.

1. 6906)10359(1
6906
3453)6906(2
6906

∴ H. C. F. is 3453.

2. 1908)2736(1
1908
828)1908(2
1656
252)828(3
756
72)252(3
216
36)72(2
72

∴ H. C. F. is 36.

3. 49608)169416(3
148824
20592)49608(2
41184
8424)20592(2
16848
3744)8424(2
7488
936)3744(4
3744

∴ H. C. F. is 936.

```
4. 40115)126025(3
         120345
         ------
           5680)40115(7
                39760
                -----
                  355)5680(16
                      355
                      ----
                      2130
∴ H. C. F. is 355.    2130
                      ----
```

```
5. 1581227)16758766(10
           15812270
           --------
             946496)1581227(1
                    946496
                    ------
                    634731)946496(1
                           634731
                           ------
                           311765)634731(2
                                  623530
                                  ------
                                   11201)311765(27
                                         22402
                                         -----
                                          87745
                                          78407
                                          -----
                                           9338)11201(1
                                                9338
                                                ----
                                                1863)9338(5
                                                     9315
                                                     ----
                                                       23)1863(81
                                                          184
                                                          ---
                                                           23
∴ H. C. F. is 23.                                          23
                                                           --
```

6. 35175)236845(6
211050

25795)35175(1
25795

9380)25795(2
18760

7035)9380(1
7035

2345)7035(3
7035

∴ H. C. F. is 2345.

XXXVII.

1. $x^2+7x+12)x^2+9x+20(1$
$x^2+7x+12$

$2x+8$

Divide by 2; $x+4)x^2+7x+12(x+3$
x^2+4x

$3x+12$
$3x+12$

2. $x^2+12x+20)x^2+14x+40(1$
$x^2+12x+20$

$2x+20$

Divide by 2; $x+10)x^2+12x+20(x+2$
x^2+10x

$2x+20$
$2x+20$

3. $x^2-13x+42)\,x^2-17x+70\,(1$

$$\begin{array}{r} x^2-13x+42 \\ \hline -4x+28 \end{array}$$

Divide by 4 and change signs ; $x-7)\,x^2-13x+42\,(x-6$

$$\begin{array}{r} x^2-7x \\ \hline -6x+42 \\ -6x+42 \\ \hline \end{array}$$

4. $x^2+5x-84)\,x^2+21x+108\,(1$

$$\begin{array}{r} x^2+\ 5x-\ 84 \\ \hline 16x+192 \end{array}$$

Divide by 16 ; $x+12)\,x^2+5x-84\,(x-7$

$$\begin{array}{r} x^2+12x \\ \hline -7x-84 \\ -7x-84 \\ \hline \end{array}$$

5. $x^2+x-12)\,x^2-2x-3\,(1$

$$\begin{array}{r} x^2+x-12 \\ \hline -3x+9 \end{array}$$

Change signs and divide by 3 ; $x-3)\,x^2+x-12\,(x+4$

$$\begin{array}{r} x^2-3x \\ \hline 4x-12 \\ 4x-12 \\ \hline \end{array}$$

6. $x^2+5xy+6y^2)\,x^2+6xy+9y^2\,(1$

$$\begin{array}{r} x^2+5xy+6y^2 \\ \hline xy+3y^2 \end{array}$$

Divide by y ; $x+3y)\,x^2+5xy+6y^2\,(x+2y$

$$\begin{array}{r} x^2+3xy \\ \hline 2xy+6y^2 \\ 2xy+6y^2 \\ \hline \end{array}$$

7. $x^2 - 6xy + 8y^2)\, x^2 - 8xy + 16y^2\, (1$

$$x^2 - 6xy + 8y^2$$
$$\overline{ - 2xy + 8y^2}$$

Divide by $2y$ and change signs ; $x - 4y)\, x^2 - 6xy + 8y^2\, (x - 2y$

$$x^2 - 4xy$$
$$\overline{- 2xy + 8y^2}$$
$$\underline{- 2xy + 8y^2}$$

8. $x^2 - 13xy - 30y^2)\, x^2 - 18xy + 45y^2\, (1$

$$x^2 - 13xy - 30y^2$$
$$\overline{ - 5xy + 75y^2}$$

Divide by $5y$ and change signs ; $x - 15y)\, x^2 - 13xy - 30y^2\, (x + 2y$

$$x^2 - 15xy$$
$$\overline{2xy - 30y^2}$$
$$\underline{2xy - 30y^2}$$

9. $x^2 - 2xy + y^2)\, x^3 - y^3\, (x + 2y$

$$x^3 - 2x^2y + xy^2$$
$$\overline{2x^2y - xy^2 - y^3}$$
$$2x^2y - 4xy^2 + 2y^3$$
$$\overline{3xy^2 - 3y^3}$$

Divide by $3y^2$; $x - y)\, x^2 - 2xy + y^2\, (x - y.$

10. $x^3 + y^3)\, x^3 + 3x^2y + 3xy^2 + y^3\, (1$

$$x^3 \qquad\qquad\qquad + y^3$$
$$\overline{3x^2y + 3xy^2}$$

Divide by $3xy$; $x + y)\, x^3 + y^3\, (x^2 - xy + y^2.$

11. $x^2-2xy+y^2)x^4-y^4(x^2+2xy+3y^2$

$$x^4-2x^3y+x^2y^2$$
$$2x^3y-x^2y^2-y^4$$
$$2x^3y-4x^2y^2+2xy^3$$
$$3x^2y^2-2xy^3-y^4$$
$$3x^2y^2-6xy^3+3y^4$$
$$4xy^3-4y^4$$

Divide by $4y^3$; $x-y)x^2-2xy+y^2(x-y$.

12. $x^3+y^3)x^5+y^5(x^2$

$$x^5+x^2y^3$$
$$-x^2y^3+y^5$$

Divide by $-y^3$; $x^2-y^2)x^3+y^3(x$

$$x^3-xy^2$$
$$xy^2+y^3$$

Divide by y^2; $x+y)x^2-y^2(x-y$.

13. $x^2+2xy+y^2)x^4-y^4(x^2-2xy+3y^2$

$$x^4+2x^3y+x^2y^2$$
$$-2x^3y-x^2y^2-y^4$$
$$-2x^3y-4x^2y^2-2xy^3$$
$$3x^2y^2+2xy^3-y^4$$
$$3x^2y^2+6xy^3+3y^4$$
$$-4xy^3-4y^4$$

Divide by $-4y^3$; $x+y)x^2+2xy+y^2(x+y$.

14. $a^2-b^2+2bc-c^2)a^2+2ab+b^2-2ac-2bc+c^2(1$

$$a^2-b^2+2bc-c^2$$
$$2ab+2b^2-2ac-4bc+2c^2$$

Divide by 2; $ab+b^2-ac-2bc+c^2=(ab-ac)+(b^2-2bc+c^2)$

$=a(b-c)+(b-c)^2$.

Divide by $b-c$; $a+b-c)a^2-b^2+2bc-c^2(a-b+c$

15. **Multiply the second expression by 3 ;**

$$12x^2+7xy+y^2)\,84x^2+9xy-3y^2\,(7$$
$$\underline{84x^2+49xy+7y^2}$$
$$-40xy-10y^2$$

Divide by $-10y$ **;** $4x+y)\,12x^2+7xy+y^2\,(3x+y.$

16. **Multiply the second expression by 2 ;**

$$6x^2+xy-y^2)\,78x^2-44xy+6y^2\,(13$$
$$\underline{78x^2+13xy-13y^2}$$
$$-57xy+19y^2$$

Divide by $-19y$ **;** $3x-y)\,6x^2+xy-y^2\,(2x+y.$

17. **Multiply the second expression by 3 ;**

$$15x^2-8xy+y^2)\,120x^2-9xy-3y^2\,(8$$
$$\underline{120x^2-64xy+8y^2}$$
$$55xy-11y^2$$

Divide by $11y$ **;** $5x-y)\,15x^2-8xy+y^2\,(3x-y.$

18. $x^4+x^3-4x^2+x+1)\,x^5-5x^3+5x^2-1\,(x-1$

$$\underline{x^5+x^4-4x^3+x^2+x}$$
$$-x^4-x^3+4x^2-x-1$$
$$\underline{-x^4-x^3+4x^2-x-1}$$

19. $x^4+4x^2+16)\,x^5+x^4-2x^3+17x^2-10x+20\,(x+1$

$$\underline{x^5+4x^3+16x}$$
$$x^4-6x^3+17x^2-26x+20$$
$$\underline{x^4+4x^2+16}$$
$$-6x^3+13x^2-26x+4$$

Change the signs of the remainder, and multiply the divisor by 6;

$$6x^3-13x^2+26x-4)\,6x^4+24x^2+96\,(x$$
$$\underline{6x^4-13x^3+26x^2-4x}$$
$$13x^3-2x^2+4x+96$$

Multiply the remainder by 6, and continue the division;

$$78x^3-12x^2+24x+576(13$$
$$78x^3-169x^2+338x-52$$
$$157x^2-314x+628$$

Divide by 157; $x^2-2x+4)6x^3-13x^2+26x-4(6x-1$

$$6x^3-12x^2+24x$$
$$-x^2+2x-4$$
$$-x^2+2x-4$$

20. $x^4+x^2y^2+y^4)x^4+2x^3y+3x^2y^2+2xy^3+y^4(1$

$$x^4\quad+x^2y^2\quad+y^4$$
$$2x^3y+2x^2y^2+2xy^3$$

Divide by $2xy$; $x^2+xy+y^2)x^4+x^2y^2+y^4(x^2-xy+y^2$.

21. $x^6-6x^4+9x^2-4)x^6+x^5-2x^4+3x^2-x-2(1$

$$x^6\quad-6x^4+9x^2\quad-4$$
$$x^5+4x^4-6x^2-x+2$$

$x^5+4x^4-6x^2-x+2)x^6-6x^4+9x^2-4(x-4$

$$x^6+4x^5-6x^3-x^2+2x$$
$$-4x^5-6x^4+6x^3+10x^2-2x-4$$
$$-4x^5-16x^4+24x^2+4x-8$$
$$10x^4+6x^3-14x^2-6x+4$$

Divide by 2, and multiply the divisor by 5;

$$5x^4+3x^3-7x^2-3x+2)5x^5+20x^4-30x^2-5x+10(x$$
$$5x^5+3x^4-7x^3-3x^2+2x$$
$$17x^4+7x^3-27x^2-7x+10$$

Multiply by 5, and continue the division;

$$85x^4+35x^3-135x^2-35x+50(17$$
$$85x^4+51x^3-119x^2-51x+34$$
$$-16x^3-16x^2+16x+16$$

Divide by -16; $x^3+x^2-x-1)5x^4+3x^3-7x^2-3x+2(5x-2$

$$\begin{array}{r} 5x^4+5x^3-5x^2-5x \\ \hline -2x^3-2x^2+2x+2 \\ -2x^3-2x^2+2x+2 \\ \hline \end{array}$$

22. Multiply the first expression by 2;

$$6a^3+19a^2b+8ab^2-5b^3)30a^4+20a^3b+8a^2b^2+12ab^3-6b^4(5a$$
$$\begin{array}{r} 30a^4+95a^3b+40a^2b^2-25ab^3 \\ \hline -75a^3b-32a^2b^2+37ab^3-6b^4 \end{array}$$

Change the signs, multiply by 2, divide by b, and continue the division;

$$6a^3+19a^2b+8ab^2-5b^3)150a^3+64a^2b-74ab^2+12b^3(25$$
$$\begin{array}{r} 150a^3+475a^2b+200ab^2-125b^3 \\ \hline -411a^2b-274ab^2+137b^3 \end{array}$$

Divide by $-137b$; $3a^2+2ab-b^2)6a^3+19a^2b+8ab^2-5b^3(2a+5b$.

23. Multiply the second expression by 5;

$$15x^3-14x^2y+24xy^2-7y^3)135x^3+165x^2y-100xy^2+10y^3(9$$
$$\begin{array}{r} 135x^3-126x^2y+216xy^2-63y^3 \\ \hline 291x^2y-316xy^2+73y^3 \end{array}$$

Divide by y; and multiply the divisor by 97;

$$291x^2-316xy+73y^2)1455x^3-1358x^2y+2328xy^2-679y^3(5x$$
$$\begin{array}{r} 1455x^3-1580x^2y+365xy^2 \\ \hline 222x^2y+1963xy^2-679y^3 \end{array}$$

Divide by y; multiply by 97, and continue the division;

$$\begin{array}{r} 21534x^2+190411xy-65863y^2(74 \\ 21534x^2-\ 23384xy+\ 5402y^2 \\ \hline 213795xy-71265y^2 \end{array}$$

Divide by $71265y$; $3x-y)291x^2-316xy+73y^2(97x-73y$.

24. Multiply the second expression by 7;

$$21x^2-83xy-27x+22y^2+99y)\,84x^2-245xy-42x-231y^2+154y\,(4$$
$$\underline{84x^2-332xy-108x+88y^2+396y}$$
$$87xy+66x-319y^2-242y$$

The remainder $=87xy-319y^2+66x-242y$
$=29y(3x-11y)+22(3x-11y)$
$=(29y+22)(3x-11y)$

Rejecting $29y+22$, which is clearly not a factor of the divisor;

$$3x-11y)\,21x^2-83xy-27x+22y^2+99y\,(7x-2y-9.$$

25. $3a^3-12a^2-a^2b+10ab-2b^2=(3a^3-a^2b)-(12a^2-10ab+2b^2)$
$=a^2(3a-b)-2(6a^2-5ab+b^2)$
$=a^2(3a-b)-2(3a-b)(2a-b)$
$=(3a-b)(a^2-4a+2b)$

Rejecting $a^2-4a+2b$, which is clearly not a factor of the second expression, we find the H. C. F. to be $3a-b$.

26. $60a^2-75ax+15x^2=15(4a^2-5ax+x^2)=3\times 5(a-x)(4a-x)$

Of these four factors, 5 and $4a-x$ are clearly not factors of the first expression: but 3 is a factor of it: divide it by 3, and divide the result by $a-x$.

$$a-x)\,6a^3-6a^2x+2ax^2-2x^3\,(6a^2+2x^2$$

∴ H. C. F. is $3(a-x)$.

27. Divide the first expression by x, and multiply the result by 2;

$$6x^2-x-2)\,42x^2-52x+16\,(7$$
$$\underline{42x^2-7x-14}$$
$$-45x+30$$

Divide by -15; $3x-2)\,6x^2-x-2\,(2x+1.$

28. $3x^3-15ax^2+a^2x-5a^3)\,6x^4+29a^2x^2+9a^4\,(2x+10a$

$$\underline{6x^4-30ax^3+2a^2x^2-10a^3x}$$
$$30ax^3+27a^2x^2+10a^3x+9a^4$$
$$\underline{30ax^3-150a^2x^2+10a^3x-50a^4}$$
$$177a^2x^2\qquad\qquad+59a^4$$

Divide by $59a^2$; $3x^2+a^2)\,3x^3-15ax^2+a^2x-5a^3\,(x-5a.$

29. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$

$$x^8 + x^6y^2 + x^2y + y^3 = x^6(x^2 + y^2) + y(x^2 + y^2)$$

$\therefore\ x^2 + y^2$ is the H. C. F.

30. Divide the first expression by 2 ;

$$x^3 + x^2 + 7x + 39)\,x^3 + 5x^2 + 7x + 3\,(1$$
$$x^3 + x^2 + 7x + 39$$
$$\overline{4x^2 - 36}$$

Now $4x^2 - 36 = 4(x^2 - 9) = 4(x + 3)(x - 3)$; and rejecting 4 and $x - 3$, we have

$$x + 3)\,x^3 + x^2 + 7x + 39\,(x^2 - 2x + 13.$$

31. $45a^3x + 3a^2x^2 - 9ax^3 + 6x^4 = x(45a^3 + 3a^2x - 9ax^2 + 6x^3)$

$18a^2x - 8x^3 = 2x(9a^2 - 4x^2)$

Reserving the common factor x ;

$$9a^2 - 4x^2)\,45a^3 + 3a^2x - 9ax^2 + 6x^3\,(5a$$
$$45a^3 - 20ax^2$$
$$\overline{3a^2x + 11ax^2 + 6x^3}$$

Divide by x, multiply by 3, and proceed with the division ;

$$9a^2 + 33ax + 18x^2\,(1$$
$$9a^2 - 4x^2$$
$$\overline{33ax + 22x^2}$$

Divide by $11x$; $3a + 2x)\,9a^2 - 4x^2\,(3a - 2x$

$\therefore$ the H. C. F. is $(3a + 2x)x$.

XXXVIII.

1. $x^2 + 5x + 6 = (x + 2)(x + 3)$; $x^2 + 7x + 10 = (x + 2)(x + 5)$; $x^2 + 12x + 20 = (x + 2)(x + 10)$.

2. $x^3 + 4x^2 - 5)\,x^3 - 3x + 2\,(1$

$$x^3 + 4x^2 - 5$$
$$\overline{-4x^2 - 3x + 7}$$

Change signs and multiply the divisor by 4 ;

$$4x^2+3x-7)\,4x^3+16x^2-20\,(x$$
$$4x^3+3x^2-7x$$
$$\overline{13x^2+7x-20}$$

Multiply by 4, and continue the division ;

$$52x^2+28x-80\,(13$$
$$52x^2+39x-91$$
$$\overline{-11x+11}$$

Change signs and divide by 11 ;

$$x-1)\,4x^2+3x-7\,(4x+7$$

Hence $x-1$ is the H. C. F. of the first two expressions ; and since $x-1$ divides x^3+4x^2-8x+3 exactly, it is the H. C. F. required.

3. $2x^2+x-1=(2x-1)(x+1)$; $x^2+5x+4=(x+4)(x+1)$;
$x^3+1=(x+1)(x^2-x+1)$.

4. $y^3-y^2-y+1=(y^3-y^2)-(y-1)=y^2(y-1)-(y-1)=(y^2-1)(y-1)$
$3y^2-2y-1=(3y+1)(y-1)$
$y^3-y^2+y-1=y^2(y-1)+(y-1)=(y^2+1)(y-1)$.

5. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$

$$x^3-4x^2+9x-10$$
$$\overline{6x^2-12x+30}$$

Divide by 6 ; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$

$$x^3-2x^2+5x$$
$$\overline{-2x^2+4x-10}$$
$$-2x^2+4x-10$$

Hence x^2-2x+5 is the H. C. F. of the first two expressions, and as it also divides $x^3+5x^2-9x+35$ exactly, it is the H. C. F. required.

6. $x^3-7x^2+16x-12)\,3x^3-14x^2+16x\,(3$

$$3x^3-21x^2+48x-36$$

$$7x^2-32x+36$$

Multiply the divisor by 7;

$$7x^2-32x+36)\,7x^3-49x^2+112x-84\,(x$$

$$7x^3-32x^2+36x$$

$$-17x^2+76x-84$$

Multiply by -7, and continue the division;

$$119x^2-532x+588\,(17$$

$$119x^2-544x+612$$

$$12x-24$$

Divide by 12; $x-2)\,7x^2-32x+36\,(7x-18$

Hence $x-2$, which also divides $5x^3-10x^2+7x-14$, is the H. C. F.

7. $y^3-5y^2+11y-15)\,y^3-y^2+3y+5\,(1$

$$y^3-5y^2+11y-15$$

$$4y^2-8y+20$$

Divide by 4; $y^2-2y+5)\,y^3-5y^2+11y-15\,(y-3$

$$y^3-2y^2+5y$$

$$-3y^2+6y-15$$

$$-3y^2+6y-15$$

Hence as y^2-2y+5 divides $2y^3-7y^2+16y-15$ exactly, it is the H. C. F.

XXXIX.

11. $\dfrac{a^2}{a^2+ab}=\dfrac{a^2}{a(a+b)}=\dfrac{a}{a+b}.$

12. $\dfrac{14m^2x}{21m^3p-7mx}=\dfrac{14m^2x}{7m(3m^2p-x)}=\dfrac{2mx}{3m^2p-x}.$

13. $\dfrac{xy}{3xy^2-5x^2yz}=\dfrac{xy}{xy(3y-5xz)}=\dfrac{1}{3y-5xz}.$

14. $\dfrac{4ax+2x^2}{8ax^3-2x^2}=\dfrac{2x(2a+x)}{2x(4ax^2-x)}=\dfrac{2a+x}{4ax^2-x}.$

15. $\dfrac{ay+y^2}{abc+bcy}=\dfrac{y(a+y)}{bc(a+y)}=\dfrac{y}{bc}.$

16. $\dfrac{4a^2x+6a^2y}{8x^2-18y^2}=\dfrac{2a^2(2x+3y)}{2(2x+3y)\,(2x-3y)}=\dfrac{a^2}{2x-3y}.$

17. $\dfrac{12ab^2-6ab}{8b^2c-2c}=\dfrac{6ab(2b-1)}{2c(2b+1)\,(2b-1)}=\dfrac{3ab}{2bc+c}.$

18. $\dfrac{c^2-4a^2}{c^2+4ac+4a^2}=\dfrac{(c+2a)\,(c-2a)}{(c+2a)\,(c+2a)}=\dfrac{c-2a}{c+2a}.$

19. $\dfrac{3x^4+3x^2y^2}{5x^4+5x^2y^2}=\dfrac{3x^2(x^2+y^2)}{5x^2(x^2+y^2)}=\dfrac{3}{5}.$

20. $\dfrac{10x-10y}{4x^2-8xy+4y^2}=\dfrac{10(x-y)}{4(x-y)\,(x-y)}=\dfrac{5}{2x-2y}.$

21. $\dfrac{ax+by}{7a^2x^2-7b^2y^2}=\dfrac{ax+by}{7(ax+by)\,(ax-by)}=\dfrac{1}{7ax-7by}.$

22. $\dfrac{6ab+8cd}{27a^2b^2x-48c^2d^2x}=\dfrac{2(3ab+4cd)}{3x(3ab+4cd)(3ab-4cd)}=\dfrac{2}{9abx-12cdx}.$

23. $\dfrac{xy-xyz}{2az-2az^2}=\dfrac{xy(1-z)}{2az(1-z)}=\dfrac{xy}{2az}.$

24. $\dfrac{7ab^3x^8-7ab^3y^2}{14a^3bcx^8-14a^3bcy^2}=\dfrac{7ab^3(x^8-y^2)}{14a^3bc(x^8-y^2)}=\dfrac{b^2}{2a^2c}.$

25. $\dfrac{5x^9+45dx^2}{10cx^9+90cdx^2}=\dfrac{5x^2(x^7+9d)}{10cx^2(x^7+9d)}=\dfrac{1}{2c}.$

26. $\dfrac{10a^2+20ab+10b^2}{5a^3+5a^2b}=\dfrac{10(a+b)\,(a+b)}{5a^2(a+b)}=\dfrac{2a+2b}{a^2}.$

27. $\dfrac{4x^2-8xy+4y^2}{48(x-y)^2}=\dfrac{4(x-y)^2}{48(x-y)^2}=\dfrac{1}{12}.$

28. $\dfrac{3mx+5nx^2}{3my+5nxy}=\dfrac{x(3m+5nx)}{y(3m+5nx)}=\dfrac{x}{y}.$

XL.

1. $\dfrac{a^2+7a+10}{a^2+5a+6}=\dfrac{(a+5)(a+2)}{(a+3)(a+2)}=\dfrac{a+5}{a+3}.$

2. $\dfrac{x^2-9x+20}{x^2-7x+12}=\dfrac{(x-4)(x-5)}{(x-4)(x-3)}=\dfrac{x-5}{x-3}.$

3. $\dfrac{x^2-2x-3}{x^2-10x+21}=\dfrac{(x-3)(x+1)}{(x-3)(x-7)}=\dfrac{x+1}{x-7}.$

4. $\dfrac{x^2-18xy+45y^2}{x^2-8xy-105y^2}=\dfrac{(x-3y)(x-15y)}{(x+7y)(x-15y)}=\dfrac{x-3y}{x+7y}.$

5. $\dfrac{x^4+x^2+1}{x^2+x+1}=\dfrac{(x^2+x+1)(x^2-x+1)}{x^2+x+1}=x^2-x+1.$

6. $\dfrac{x^6+2x^3y^3+y^6}{x^6-y^6}=\dfrac{(x^3+y^3)(x^3+y^3)}{(x^3+y^3)(x^3-y^3)}=\dfrac{x^3+y^3}{x^3-y^3}.$

7. $x^3-4x^2+9x-10)x^3+2x^2-3x+20(1$

$$\begin{array}{r} x^3-4x^2+9x-10 \\ \hline 6x^2-12x+30 \end{array}$$

Divide by 6; $x^2-2x+5)x^3-4x^2+9x-10(x-2$

Hence x^2-2x+5 is the H. C. F.

8. $x^3-5x^2+11x-15)x^3-x^2+3x+5(1$

$$\begin{array}{r} x^3-5x^2+11x-15 \\ \hline 4x^2-8x+20 \end{array}$$

Divide by 4; $x^2-2x+5)x^3-5x^2+11x-15(x-3$

Hence x^2-2x+5 is the H. C. F.

9. $x^3-8x^2+21x-18)3x^3-16x^2+21x(3$

$$\begin{array}{r} 3x^3-24x^2+63x-54 \\ \hline 8x^2-42x+54 \end{array}$$

Divide by 2; $4x^2-21x+27$

Multiply divisor by 4 ; $4x^2 - 21x + 27)\,4x^3 - 32x^2 + 84x - 72\,(x$

$$4x^3 - 21x^2 + 27x$$
$$-11x^2 + 57x - 72$$

Multiply by -4, and continue the division ;

$$4x^2 - 21x + 27)\,44x^2 - 228x + 288\,(11$$
$$44x^2 - 231x + 297$$
$$3x - 9$$

Divide by 3 ; $x - 3)\,4x^2 - 21x + 27\,(4x - 9$

Hence $x - 3$ is the H. C. F.

10. The H. C. F. is $x - 2$, see the work in xxxviii. 6.

11. $x^4 - x^3y - xy^3 - y^4)\,x^4 + x^3y + xy^3 - y^4\,(1$

$$x^4 - x^3y - xy^3 - y^4$$
$$2x^3y + 2xy^3$$

Divide by $2xy$; $x^2 + y^2)\,x^4 - x^3y - xy^3 - y^4\,(x^2 - xy - y^2.$

12. $a^3 - 3a + 2)\,a^3 + 4a^2 - 5\,(1$

$$a^3 - 3a + 2$$
$$4a^2 + 3a - 7$$

Multiply divisor by 4 ; $4a^2 + 3a - 7)\,4a^3 - 12a + 8\,(a$

$$4a^3 + 3a^2 - 7a$$
$$-3a^2 - 5a + 8$$

Multiply by -4 and continue the division ;

$$12a^2 + 20a - 32\,(3$$
$$12a^2 + 9a - 21$$
$$11a - 11$$

Divide by 11 ; $a - 1)\,4a^2 + 3a - 7\,(4a + 7.$

13. $b^3 - 6b + 5)\,b^3 + 4b^2 - 5b\,(1$

$$b^3 - 6b + 5$$
$$4b^2 + b - 5$$

Multiply divisor by 4; $4b^2+b-5)\,4b^3-24b+20\,(b$

$$\begin{array}{r} 4b^3+b^2-5b \\ \hline -b^2-19b+20 \end{array}$$

Change signs; $b^2+19b-20)\,4b^2+b-5\,(4$

$$\begin{array}{r} 4b^2+76b-80 \\ \hline -75b+75 \end{array}$$

Hence the H. C. F. is found to be $b-1$.

14. $m^3-7m+6)\,m^3+3m^2-4m\,(1$

$$\begin{array}{r} m^3-7m+6 \\ \hline 3m^2+3m-6 \end{array}$$

Divide by 3; $m^2+m-2)\,m^3-7m+6\,(m-1$

$$\begin{array}{r} m^3+m^2-2m \\ \hline -m^2-5m+6 \\ -m^2-\ m+2 \\ \hline -4m+4 \end{array}$$

Hence the H, C. F. is found to be $m-1$.

15. $a^3+1)\,a^3+2a^2+2a+1\,(1$

$$\begin{array}{r} a^3 \qquad\quad +1 \\ \hline 2a^2+2a \end{array}$$

Divide by $2a$, and H. C. F. is found to be $a+1$.

16. $$\frac{3ax^2-13ax+14a}{7x^3-17x^2+6x}=\frac{a(3x^2-13x+14)}{x(7x^2-17x+6)}$$

$3x^2-13x+14)\,21x^2-51x+18\,(7$

$$\begin{array}{r} 21x^2-91x+98 \\ \hline 40x-80 \end{array}$$

Whence H. C. F. of $3x^2-13x+14$ and $7x^2-17x+6$ is found to be $x-2$.

17. $$\frac{14x^2-34x+12}{9ax^2-39ax+42a}=\frac{2(7x^2-17x+6)}{3a(3x^2-13x+14)};$$

and as in the preceding question, the H. C. F. is found to be $x-2$.

18. $\dfrac{10a-24a^2+14a^3}{15-24a+3a^2+6a^3}=\dfrac{2a(5-12a+7a^2)}{3(5-8a+a^2+2a^3)}$

$$\begin{array}{r}5-12a+7a^2)\,5-8a+a^2+2a^3\,(1\\ 5-12a+7a^2\\ \hline 4a-6a^2+2a^3\end{array}$$

Divide by $2a$; $2-3a+a^2=(1-a)(1-2a)$

Hence the H. C. F. is found to be $1-a$.

19. $\dfrac{2ab^3+ab^2-8ab+5a}{7b^3-12b^2+5b}=\dfrac{a(2b^3+b^2-8b+5)}{b(7b^2-12b+5)}$

$$\begin{array}{r}7b^2-12b+5)\,14b^3+7b^2-56b+35\,(2b\\ 14b^3-24b^2+10b\\ \hline 31b^2-66b+35\\ 7\\ \hline 217b^2-462b+245\,(31\\ 217b^2-372b+155\\ \hline -90b+90\end{array}$$

Hence the H. C. F. is found to be $b-1$.

20. $$\begin{array}{r}a^3-4a^2+6a-4)\,a^3-3a^2+3a-2\,(1\\ a^3-4a^2+6a-4\\ \hline a^2-3a+2)\,a^3-4a^2+6a-4\,(a-1\\ a^3-3a^2+2a\\ \hline -a^2+4a-4\\ -a^2+3a-2\\ \hline a-2\end{array}$$

Hence we find $a-2$ to be the H. C. F.

21. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}=\dfrac{(3x-1)(x+1)}{(x^2-1)(x+1)}=\dfrac{3x-1}{x^2-1}$.

22. $\dfrac{a^2-a-20}{a^2+a-12}=\dfrac{(a-5)(a+4)}{(a-3)(a+4)}=\dfrac{a-5}{a-3}$.

23. $x^3-3x^2+4x-2)\,x^3-x^2-2x+2\,(1$

$$x^3-3x^2+4x-2$$

$$2x^2-6x+4$$

Divide by 2 ; $x^2-3x+2)\,x^3-3x^2+4x-2\,(x$

$$x^3-3x^2+2x$$

$$2x-2$$

Hence we find $x-1$ to be the H. C. F.

24. $$\frac{x^2+y^2+z^2+2xy+2xz+2yz+z^2-2yz+y^2+x^2-2xz+z^2+y^2-2xy+x^2}{x^2+y^2+z^2}$$

$$=\frac{3x^2+3y^2+3z^2}{x^2+y^2+z^2}=3.$$

25. Multiply the numerator by 7 ;

$7x^3-19x^2+17x-5)\,14x^4-7x^3-63x^2+91x-35\,(2x$

$$14x^3-38x^3+34x^2-10x$$

$$31x^3-97x^2+101x-35$$

$$7$$

$$217x^3-679x^2+707x-245\,(31$$

$$217x^3-589x^2+527x-155$$

$$-90x^2+180x-90$$

Divide by -90 ; $x^2-2x+1)\,7x^3-19x^2+17x-5\,(7x-5$

$$7x^3-14x^2+7x$$

$$-5x^2+10x-5$$

$$-5x^2+10x-5$$

Hence x^2-2x+1 is the H. C. F.

26. $8x^4-30x^3+31x^2-12)\,16x^4-53x^2+45x+6\,(2$

$$16x^4-60x^3+62x^2-24$$

$$60x^3-115x^2+45x+30$$

Divide by 5, and multiply the divisor by 3;

$$12x^3-23x^2+9x+6)\,24x^4-90x^3+93x^2-36\,(2x$$
$$\underline{12x^4-46x^3+18x^2+12x}$$
$$-44x^3+75x^2-12x-36$$

Multiply by -3, and continue the division;

$$132x^3-225x^2+36x+108\,(11$$
$$\underline{132x^3-253x^2+99x+\ \ 66}$$
$$28x^2-63x+\ \ 42$$

Divide by 7; $4x^2-9x+6)\,12x^3-23x^2+9x+6\,(3x+1$

Hence $4x^2-9x+6$ is the H. C. F.

27. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}=\dfrac{(2x-3a)(2x-3a)}{(2x-3a)(4x^2+6ax+9a^2)}$.

28. $6x^3-23x^2+16x-3)\,6x^3-17x^2+11x-2\,(1$
$$\underline{6x^3-23x^2+16x-3}$$
$$6x^2-\ \ 5x+1$$

$6x^2-5x+1)\,6x^3-23x^2+16x-3\,(x-3$

Hence $6x^2-5x+1$ is the H. C. F.

29. $x^3-6x^2+11x-6)\,x^3-2x^2-\ \ x\ \ +2\,(1$
$$\underline{x^3-6x^2+11x-6}$$
$$4x^2-12x+8$$

Divide by 4; $x^2-3x+2)\,x^3-6x^2+11x-6\,(x-3$

Hence x^2-3x+2 is the H. C. F.

30. $m^3+m^2+m-3)\,m^3+3m^2+5m+3\,(1$
$$\underline{m^3+\ \ m^2+\ \ m-3}$$
$$2m^2+4m+6$$

Divide by 2; $m^2+2m+3)\,m^3+m^2+m-3\,(m-1$

Hence m^2+2m+3 is the H. C. F.

31. $\dfrac{x^5+5x^4-x^2-5x}{x^4+3x^3-x-3}=\dfrac{x^4(x+5)-x(x+5)}{x^3(x+3)-(x+3)}=\dfrac{(x+5)(x^3-1)x}{(x+3)(x^3-1)}=\dfrac{x^2+5x}{x+3}$.

32. $$\frac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}=\frac{a^2-(b^2+2bc+c^2)}{(a^2+2ab+b^2)-c^2}=\frac{a^2-(b+c)^2}{(a+b)^2-c^2}$$
$$=\frac{(a+b+c)(a-b-c)}{(a+b+c)(a+b-c)}=\frac{a-b-c}{a+b-c}.$$

33. Multiply the numerator by 3;

$9a^2+3ab-2b^2)\,45a^2+\ 3ab-\ 6b^2\,(5$

$45a^2+15ab-10b^2$

$-12ab+\ 4b^2$

Divide by $-4b$; $3a-b)\,9a^2+3ab-2b^2\,(3a+2b$

Hence $3a-b$ is the H. C. F.

34. $x^2-7x+10=(x-5)(x-2)$

$x-5$ is clearly not a factor of the denominator;

$x-2)2x^2-x-6(2x+3$

Hence $x-2$ is the H. C. F.

35. $x^3+3x^2+4x+12)\,x^3+4x^2+4x+3\,(1$

$x^3+3x^2+4x+12$

x^2-9

Divide by $x-3$; $x+3)\,x^3+3x^2+4x+12\,(x^2+4$

Hence $x+3$ is the H. C. F.

36. $2x^3-x-1)2x^4-2x^2-4x+4)x$

$2x^4-\ x^2-x$

$-\ x^2-3x+4$

Change signs; $x^2+3x-4=(x+4)(x-1)$

Divide by $x+4$; $x-1)\,2x^3-x-1\,(2x^2+2x+1$

Hence $x-1$ is the H. C. F.

37. $3x^2-4x-15)\,3x^3-6x^2-45x+108\,(x$

$3x^3-4x^2-15x$

$-2x^2-30x+108$

Divide by -2; $x^2+15x-54=(x+18)(x-3)$

Divide by $x+18$; $x-3)\,3x^2-4x-15\,(3x+5$

Hence $x-3$ is the H. C. F.

38. $3x^3+x^2-5x+21)\,6x^3+29x^2+26x-21\,(2$

$$6x^3+\ \ 2x^2-10x+42$$
$$27x^2+36x-63$$

Divide by 9; $3x^2+4x-7)\,3x^3+\ x^2-5x+21\,(x-1$

$$3x^3+4x^2-7x$$
$$-3x^2+2x+21$$
$$-3x^2-4x+\ 7$$
$$6x+14$$

Divide by 2; $3x+7)\,3x^2+4x-7\,(x-1$

Hence $3x+7$ is the H. C. F.

39. $4x^3-3x^2-8x-1)\,4x^4-4x^3-16x^2-4x+4\,(x$

$$4x^4-3x^3-\ 8x^2-x$$
$$-\ x^3-\ 8x^2-3x+4$$

Change signs; $x^3+8x^2+3x-4)\,4x^3-\ 3x^2-\ 8x-\ 1\,(4$

$$4x^3+32x^2+12x-16$$
$$-35x^2-20x+15$$

Divide by -5; $7x^2+4x-3=(7x-3)(x+1)$

Divide by $7x-3$; $x+1)\,x^3+8x^2+3x-4\,(x^2+7x-4$

Hence $x+1$ is the H. C. F.

40. $a^3-7a^2+16a-12)\,3a^3-14a^2+16a\,(3$

$$3a^3-21a^2+48a-36$$
$$7a^2-32a+36$$

$7a^2-32a+36)\,7a^3-49a^2+112a-84\,(a$

$$7a^3-32a^2+\ 36a$$
$$-17a^2+\ 76a-84$$
$$7$$
$$-119a^2+532a-588\,(-17$$
$$-119a^2+544a-612$$
$$-12a+24$$

Hence we find $a-2$ to be the H. C. F.

XLII.

1. $\dfrac{a-b}{a^2+ab} \times \dfrac{a^2-b^2}{a^2-ab} = \dfrac{(a-b)(a+b)(a-b)}{a(a+b)a(a-b)} = \text{etc.}$

2. $\dfrac{x^2+4x}{x^2-3x} \times \dfrac{4x^2-12x}{3x^2+12x} = \dfrac{x(x+4)4x(x-3)}{x(x-3)3x(x+4)} = \text{etc.}$

3. $\dfrac{x^2+3x+2}{x^2-5x+6} \times \dfrac{x^2-7x+12}{x^2+x} = \dfrac{(x+2)(x+1)(x-3)(x-4)}{(x-2)(x-3)x(x+1)} = \text{etc.}$

4. $\dfrac{x^2+x-2}{x^2-7x} \times \dfrac{x^2-13x+42}{x^2+2x} = \dfrac{(x+2)(x-1)(x-7)(x-6)}{x(x-7)x(x+2)} = \text{etc.}$

5. $\dfrac{x^2-11x+30}{x^2-6x+9} \times \dfrac{x^2-3x}{x^2-5x} = \dfrac{(x-6)(x-5)x(x-3)}{(x-3)(x-3)x(x-5)} = \text{etc.}$

6. $\dfrac{x^2-4}{x^2+5x} \times \dfrac{x^2-25}{x^2+2x} = \dfrac{(x+2)(x-2)(x+5)(x-5)}{x(x+5)x(x+2)} = \text{etc.}$

7. $\dfrac{(a-3)(a-1)}{(a-4)(a-1)} \times \dfrac{(a-5)(a-4)}{(a-7)(a-3)} \times \dfrac{a(a-7)}{a(a-5)} = 1.$

8. $\dfrac{(b-6)(b-1)}{(b+4)(b-1)} \times \dfrac{(b+4)(b+6)}{(b-8)(b-6)} \times \dfrac{b^2(b-8)}{b(b+6)} = b.$

9. $\dfrac{(x+y)(x-y)}{(x-y)(x-2y)} \times \dfrac{y(x-2y)}{x(x+y)} \times \dfrac{x(x-y)}{(x-y)(x-y)} = \dfrac{y}{x-y}.$

10. $\dfrac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)} \times \dfrac{(c+a-b)(c-a+b)}{(c+a+b)(c-a-b)} = \dfrac{c-a+b}{c-a-b}.$

11. $\dfrac{(x-m+n)(x-m-n)}{(x-n+m)(x-n-m)} \times \dfrac{(x+n-m)(x-n+m)}{(x+m-n)(x-m+n)} = \dfrac{x-m+n}{x+m-n}.$

12. $\dfrac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)} \times \dfrac{(a-b+d-c)(a-b-d+c)}{(a-c+d-b)(a-c-d+b)} = 1.$

13. $\dfrac{(x^2-2xy+y^2)-z^2}{(x^2+2xy+y^2)-z^2} \times \dfrac{x+y-z}{x-y+z} = \dfrac{(x-y+z)(x-y-z)(x+y-z)}{(x+y+z)(x+y-z)(x-y+z)} = \text{etc.}$

XLIII.

1. $\dfrac{2a}{x} \times \dfrac{5c}{3b} = \dfrac{10ac}{3bx}.$

2. $\dfrac{15y}{14z} \times \dfrac{7z}{5y^2} = \dfrac{3}{2y}.$

3. $\dfrac{8x^4y}{15ab^3} \times \dfrac{30ab^2}{2x^3} = \dfrac{8xy}{b}.$

4. $\dfrac{4a}{nx} \times \dfrac{1}{3ab} = \dfrac{4}{3bnx}.$

5. $\dfrac{3p}{2(p-1)} \times \dfrac{p-1}{2p} = \dfrac{3}{4}$.

6. $1 \times \dfrac{5x}{4a} = \dfrac{5x}{4a}$.

7. $\dfrac{5x}{7} \times \dfrac{1}{2} = \dfrac{5x}{14}$.

8. $\dfrac{1}{(x-1)(x-2)} \times \dfrac{x-1}{1} = \dfrac{1}{x-2}$.

9. $\dfrac{1}{(x-15)(x-2)} \times \dfrac{x-15}{1} = \dfrac{1}{x-2}$.

XLV.

1. $x^2 = x \times x$; $ax + x^2 = x(a+x)$; $\therefore$ L. C. M. is $x \times x(a+x)$.

2. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $\therefore$ L. C. M. is $(x+1)(x-1)x$.

3. $a^2 - b^2 = (a+b)(a-b)$; $a^2 + ab = a(a+b)$; $\therefore$ L. C. M. is $(a+b)(a-b)a$.

4. $4x^2 - 1 = (2x-1)(2x+1)$; $\therefore$ L. C. M. is $4x^2 - 1$.

5. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$; $\therefore$ L. C. M. is $a^3 + b^3$.

6. $x^2 - 1 = (x+1)(x-1)$; $\therefore$ L. C. M. is $x^2 - 1$.

7. $x^3 - 1 = (x-1)(x^2+x+1)$; $\therefore$ L. C. M. is $(x+1)(x^3-1)$.

8. $x^3 + 1 = (x+1)(x^2-x+1)$; $\therefore$ L. C. M. is $(x^2+1)(x^3+1)$.

9. $x^2 - 1 = (x+1)(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)(x^2+x+1)$.

10. $x^4 - 1 = (x^2+1)(x^2-1)$; $\therefore$ L. C. M. is $x^4 - 1$.

11. $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$; $x^3 + 1 = (x+1)(x^2-x+1)$;
$\therefore$ L. C. M. is $x(x-1)(x^2+x+1)(x+1)(x^2-x+1)$, etc.

12. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)x(x^2+x+1)$, etc.

13. $4a^2 - 1 = (2a+1)(2a-1)$; $8a^3 + 1 = (2a+1)(4a^2-2a+1)$;
$\therefore$ L. C. M. is $(2a+1)(2a-1)(4a^2-2a+1)$, etc.

14. $2x^2+2xy=2x(x+y)$; $\therefore$ L. C. M. is $2x^2+2xy$.

15. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is $(a+b)(a+b)(a-b)$.

16. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is a^2-b^2.

17. $2(1-x^2)=2(1+x)(1-x)$; $\therefore$ L. C. M. is $4(1+x)(1-x)$.

18. $x^3-1=(x-1)(x^2+x+1)$; $\therefore$ L. C. M. is x^3-1.

19. L. C. M. is $(a-b)(a-c)(b-c)$.

20. L. C. M. is $(x+1)(x+2)(x+3)$.

21. $x^2-y^2=(x+y)(x-y)$;
$\therefore$ L. C. M. is $(x+y)(x-y)(x+y)(x-y)$.

22. $a^2-1=(a+1)(a-1)$; $\therefore$ L. C. M. is $(a+3)(a+1)(a-1)$.

23. $x(x^2-y^2)=x(x+y)(x-y)$; $\therefore$ L. C. M. is $x^2(x-y)(x+y)$.

24. L. C. M. is $(x+1)(x+3)(x+2)(x+4)$.

25. $x^2-y^2=(x+y)(x-y)$; $12(x^3+y^3)=12(x+y)(x^2-xy+y^2)$;
$\therefore$ L. C. M. is $12(x+y)(x-y)(x-y)(x^2-xy+y^2)$.

26. $6(x^2+xy)=6x(x+y)$; $8(xy-y^2)=8y(x-y)$;
$10(x^2-y^2)=10(x+y)(x-y)$;
$\therefore$ L. C. M. is $120xy(x+y)(x-y)$.

XLVI.

1. $x^2+5x+6=(x+2)(x+3)$
$x^2+6x+8=(x+2)(x+4)$.

2. $a^2-a-20=(a-5)(a+4)$
$a^2+a-12=(a-3)(a+4)$.

3. $x^2+3x+2=(x+1)(x+2)$
$x^2+4x+3=(x+1)(x+3)$.

4. $x^2+11x+30=(x+5)(x+6)$
$x^2+12x+35=(x+5)(x+7)$.

5. $x^2-\ 9x-22=(x-11)(x+2)$
$x^2-13x+22=(x-11)(x-2)$.

6. $2x^2+3x+1=(2x+1)(x+1)$
$x^2-x-2=(x-2)(x+1)$.

7. $x^3+x^2y+xy+y^2=(x^2+y)(x+y)$
$x^4-y^4=(x^2+y^2)(x+y)(x-y)$.

8. $x^2-8x+15=(x-3)(x-5)$
$x^2+2x-15=(x-3)(x+5)$.

9. $21x^2-26x+8)\,21x^3-12x^2-63x+36\,(x$

$$\begin{array}{r}
21x^3-26x^2+8x \\
\hline
14x^2-71x+36 \\
3 \\
\hline
42x^2-213x+108\,(2 \\
42x^2-52x+16 \\
\hline
-161x+92
\end{array}$$

Divide by -23; $7x-4)\,21x^2-26x+8\,(3x-2$

$\therefore$ L. C. M. is $(3x-2)(7x^3-4x^2-21x+12)$
or $(3x-2)(7x-4)(x^2-3)$.

10. $x^3+x^2y+xy^2+y^3=(x^2+y^2)(x+y)$
$x^3-x^2y+xy^2-y^3=(x^2+y^2)(x-y)$.

11. $a^3-2a^2b-ab^2+2b^3)\,a^3+2a^2b-ab^2-2b^3\,(1$

$$\begin{array}{r}
a^3-2a^2b-ab^2+2b^3 \\
\hline
4a^2b \qquad -4b^3
\end{array}$$

Divide by $4b$; $a^2-b^2)\,a^3-2a^2b-ab^2+2b^3\,(a-2b$

$\therefore$ L. C. M. is $(a-2b)(a^3+2a^2b-ab^2-2b^3)$
or $(a-2b)(a^2-b^2)(a+2b)$.

XLVII.

1. $x^2-3x+2=(x-1)(x-2)$
$x^2-4x+3=(x-1)(x-3)$
$x^2-5x+4=(x-1)(x-4).$

2. $x^2+5x+4=(x+1)(x+4)$
$x^2+4x+3=(x+1)(x+3)$
$x^2+7x+12=(x+3)(x+4).$

3. $x^2-\ \ 9x+20=(x-4)(x-5)$
$x^2-12x+35=(x-7)(x-5)$
$x^2-11x+28=(x-4)(x-7).$

4. $6x^2-x-2)\,42x^2-34x+\ 4\,(7$

$$42x^2-\ 7x-14$$

$$-27x+18$$

Divide by -9; $\ 3x-2)\,6x^2-x-2\,(2x+1.$

Hence L. C. M. of first two expressions is

$$(2x+1)(21x^2-17x+2),\ \text{or}\ 42x^3-13x^2-13x+2.$$

We have now to find the L. C. M. of this and $14x^2+5x-1$.

$$14x^2+5x-1)\,42x^3-13x^2-13x+2\,(3x-2$$

$$42x^3+15x^2-\ 3x$$

$$-28x^2-10x+2$$

$$-28x^2-10x+2$$

$\therefore$ L. C. M. is $(3x-2)(14x^2+5x-1)$, etc.

5. $x^2-1=(x+1)(x-1)$
$x^2+2x-3=(x+3)(x-1)$
$6x^2-x-2=(3x-2)(2x+1).$

6. $x^3-27=(x-3)(x^2+3x+9)$
$x^2-15x+36=(x-3)(x-12)$
$x^3-3x^2-2x+6=x^2(x-3)-2(x-3)=(x^2-2)(x-3).$

XLVIII.

1. L. C. D. 20; new numerators $15x$, $16x$.

2. L. C. D. 18; new numerators $9x-21$, $4x-9$.

3. L. C. D. $10x^2$; new numerators $4x-8y$, $3x^2-8xy$.

4. L. C. D. $10a^2$; new numerators $20a+25b$, $6a^2-8ab$.

5. L. C. D. $60a^2c$; new numerators $48a^2-60ac$, $15a-10c$.

6. L. C. D. a^3b^2; new numerators $ab-b^2$, a^4-a^3b.

7. L. C. D. $1-x^2$; new numerators $3-3x$, $3+3x$.

8. L. C. D. $1-y^4$; new numerators $2+2y^2$, $2-2y^2$.

9. L. C. D. $1-x^2$; new numerators $5+5x$, 6.

10. L. C. D. $c(b+x)$; new numerators $ab+ax$, b.

11. L. C. D. $(a-b)(b-c)(a-c)$; new numerators $a-c$, $b-c$.

12. L. C. D. $abc(a-b)(a-c)(b-c)$; new numerators $bc-c^2$, $ab-b^2$.

XLIX.

1. $\dfrac{12x+21}{15}+\dfrac{3x-4}{15}=\dfrac{15x+17}{15}$.

2. $\dfrac{36a-48b}{84}-\dfrac{56a-28b+28c}{84}+\dfrac{91a-28c}{84}=\dfrac{71a-20b-56c}{84}$.

3. $\dfrac{24x-18y}{42}+\dfrac{9x+21y}{42}-\dfrac{10x-4y}{42}+\dfrac{9x+2y}{42}=\dfrac{32x+9y}{42}$.

4. $\dfrac{30x-20y}{50x}+\dfrac{25x-35y}{50x}+\dfrac{16x^2+4xy}{50x}=\dfrac{16x^2+55x+4xy-55y}{50x}$.

5. $\dfrac{16x^2-28y^2}{12x^2}+\dfrac{6x^2-16xy}{12x^2}+\dfrac{5x^2-2x^2y}{12x^2}=\dfrac{27x^2-2x^2y-16xy-28y^2}{12x^2}$.

6. $\dfrac{180a^2+225b^2}{90b^2}+\dfrac{54ab+36b^2}{90b^2}+\dfrac{70b^2-20ab^2}{90b^2}=\dfrac{180a^2+54ab+331b^2-20ab^2}{90b^2}.$

7. $\dfrac{80x^3+100x^2}{60x^2}-\dfrac{36x^2-84x}{60x^2}+\dfrac{45}{60x^2}=\dfrac{80x^3+64x^2+84x+45}{60x^2}.$

8. $\dfrac{70a^2+28ab}{42ac}-\dfrac{84c^2-63bc}{42ac}+\dfrac{18ab-21bc}{42ac}=\dfrac{70a^2+46ab+42bc-84c^2}{42ac}$, etc.

9. $\dfrac{2ac+5c^2}{a^2c^2}+\dfrac{4a^2c-3ac^2}{a^2c^2}-\dfrac{5ac-2c^2}{a^2c^2}=\dfrac{4a^2c-3ac^2-3ac+7c^2}{a^2c^2}$, etc.

10. $\dfrac{3x^2y^2-4xy}{x^3y^3}-\dfrac{5x^2y^2+7x^2}{x^3y^3}-\dfrac{6x^2y^2-11y^2}{x^3y^3}=\dfrac{11y^2-8x^2y^2-4xy-7x^2}{x^3y^3}.$

11. $\dfrac{abc^2-b^2c^2}{a^3b^2c^2}+\dfrac{4a^2bc-5ab^2c}{a^3b^2c^2}+\dfrac{3a^4-7a^3b}{a^3b^2c^2}=\dfrac{3a^4-7a^3b+4a^2bc-5ab^2c+abc^2-b^2c^2}{a^3b^2c^2}.$

L.

1. $\dfrac{x+5+x-6}{(x-6)(x+5)}=\dfrac{2x-1}{(x-6)(x+5)}.$

2. $\dfrac{x-3-(x-7)}{(x-7)(x-3)}=\dfrac{4}{(x-7)(x-3)}.$

3. $\dfrac{1-x+1+x}{(1+x)(1-x)}=\dfrac{2}{(1+x)(1-x)}.$

4. $\dfrac{x^2+2xy+y^2-(x^2-2xy+y^2)}{(x+y)(x-y)}=\dfrac{4xy}{(x+y)(x-y)}.$

5. $\dfrac{1+x-2}{(1+x)(1-x)}=\dfrac{x-1}{(1+x)(1-x)}=\dfrac{-(1-x)}{(1+x)(1-x)}=\dfrac{-1}{1+x}$

6. $\dfrac{a(c+dx)-(ad-bc)x}{c(c+dx)}=\dfrac{ac+bcx}{c(c+dx)}=\dfrac{a+bx}{c+dx}.$

7. $\dfrac{x^2-xy+x^2+xy}{(x+y)(x-y)}=\dfrac{2x^2}{(x+y)(x-y)}.$

8. $\frac{x-y+x}{(x-y)^2}=\frac{2x-y}{(x-y)^2}$.

9. $\frac{2(x+a)+3a}{(x+a)^2}=\frac{2x+5a}{(x+a)^2}$

10. $\frac{a-x+a+x}{2a(a+x)(a-x)}=\frac{2a}{2a(a+x)(a-x)}=\frac{1}{(a+x)(a-x)}$.

LI.

1. $\frac{1-a+1+a}{1-a^2}+\frac{2a}{1-a^2}=\frac{2+2a}{1-a^2}=\frac{2(1+a)}{1-a^2}=\frac{2}{1-a}$.

2. $\frac{1+x-(1-x)}{1-x^2}+\frac{2x}{1+x^2}=\frac{2x}{1-x^2}+\frac{2x}{1+x^2}=\frac{2x+2x^3+2x-2x^3}{1-x^4}=\frac{4x}{1-x^4}$.

3. $\frac{x+x^2-x^2}{1-x^2}+\frac{x}{1+x^2}=\frac{x}{1-x^2}+\frac{x}{1+x^2}=\frac{x+x^3+x-x^3}{1-x^4}=\frac{2x}{1-x^4}$.

4. $\frac{a+b-(a-b)}{a^2-b^2}-\frac{2b}{a^2+b^2}=\frac{2b}{a^2-b^2}-\frac{2b}{a^2+b^2}=\frac{2a^2b+2b^3-(2a^2b-2b^3)}{a^4-b^4}$

$\frac{4b^3}{a^4-b^4}-\frac{4b^3}{a^4+b^4}=\frac{4a^4b^3+4b^7-(4a^4b^3-4b^7)}{a^8-b^8}=\frac{8b^7}{a^8-b^8}$.

5. $\frac{x^2+xy+y^2}{y(x+y)}+\frac{x^2}{x(x+y)}=\frac{x^2+xy+y^2}{y(x+y)}+\frac{x}{x+y}=\frac{x^2+xy+y^2+xy}{y(x+y)}$

$=\frac{(x+y)^2}{y(x+y)}=\frac{x+y}{y}$.

6. $\frac{x^2-9+x^2-16}{(x+4)(x-3)}+\frac{x+5}{x+7}=\frac{2x^2-25}{(x+4)(x-3)}+\frac{x+5}{x+7}$

$=\frac{2x^3-25x+14x^2-175+(x+5)(x^2+x-12)}{(x+4)(x-3)(x+7)}$

$=\frac{2x^3-25x+14x^2-175+x^3+6x^2-7x-60}{(x+4)(x-3)(x+7)}$, etc.

7. $\dfrac{x^2-4x+3+x^2-4x+4}{(x-2)(x-3)}+\dfrac{x-3}{x-4}=\dfrac{2x^2-8x+7}{(x-2)(x-3)}+\dfrac{x-3}{x-4}$

$$=\frac{2x^3-16x^2+39x-28+(x-3)(x^2-5x+6)}{(x-2)(x-3)(x-4)}$$

$$=\frac{2x^3-16x^2+39x-28+x^3-8x^2+21x-18}{(x-2)(x-3)(x-4)}, \text{ etc.}$$

8. $\dfrac{3x-3a+4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}=\dfrac{(3x+a)(x-a)-5a^2}{(x-a)^3}=\dfrac{3x^2-2ax-a^2-5a^2}{(x-a)^3}$

$$=\frac{3x^2-2ax-6a^2}{(x-a)^3}.$$

9. $\dfrac{x+2-(x-1)}{(x-1)(x+2)}-\dfrac{3}{(x+1)(x+2)}=\dfrac{3}{(x-1)(x+2)}-\dfrac{3}{(x+1)(x+2)}$

$$=\frac{3x+3-(3x-3)}{(x-1)(x+1)(x+2)}=\frac{6}{(x-1)(x+1)(x+2)}.$$

10. $\dfrac{x+3}{(x+1)(x+2)(x+3)}-\dfrac{3}{(x+1)(x+2)(x+3)}=\dfrac{x}{(x+1)(x+2)(x+3)}.$

11. $\dfrac{x^2}{x^2-1}+\dfrac{x^2+x}{x^2-1}+\dfrac{x^2-x}{x^2-1}=\dfrac{3x^2}{x^2-1}.$

12. $\dfrac{a+e}{(a+c)(a+d)(a+e)}-\dfrac{a+d}{(a+c)(a+d)(a+e)}=\dfrac{e-d}{(a+c)(a+d)(a+e)}.$

13. $\dfrac{(a-b)(a+b)+(b-c)(b+c)+(c-a)(c+a)}{(b+c)(c+a)(a+b)}$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b+c)(c+a)(a+b)}=0.$$

14. $\dfrac{(x-a)^2+(x-b)^2-(a-b)^2}{(x-a)(x-b)}=\dfrac{x^2-2ax+a^2+x^2-2bx+b^2-a^2+2ab-b^2}{(x-a)(x-b)}$

$$=\frac{2(x^2-ax-bx+ab)}{x^2-ax-bx+ab}=2.$$

15. $\dfrac{(x+y)^2-2xy}{y(x+y)}+\dfrac{x^2y-x^3}{y(x^2-y^2)}=\dfrac{x^2+y^2}{y(x+y)}+\dfrac{x^2y-x^3}{y(x^2-y^2)}$

$$=\frac{(x^2+y^2)(x-y)+x^2y-x^3}{y(x^2-y^2)}=\frac{x^3+xy^2-x^2y-y^3+x^2y-x^3}{y(x^2-y^2)}$$

$$=\frac{y^2(x-y)}{y(x^2-y^2)}=\frac{y}{x+y}.$$

16. $\dfrac{(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)}{(b-c)(c-a)(a-b)}$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b-c)(c-a)(a-b)}=0.$$

17. $\dfrac{x(x-y)}{x^3-y^3}+\dfrac{2xy}{x^3-y^3}=\dfrac{x^2+xy}{x^3-y^3}.$

18. $\dfrac{2b-2c+2a-2b}{(a-b)(b-c)}+\dfrac{2}{c-a}=\dfrac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)}$;

$$\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$$

$$=\frac{(2ac-2a^2-2c^2+2ac+2ab-2b^2-2ac+2bc)+(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2)}{(a-b)(b-c)(c-a)}$$

$=0.$

19. $\dfrac{(a+b)^2-2ab}{b(a+b)}+\dfrac{a^2b-a^3}{b(a^2-b^2)}=\dfrac{(a^2+b^2)(a-b)+a^2b-a^3}{b(a^2-b^2)}$

$$=\frac{a^3+ab^2-a^2b-b^3+a^2b-a^3}{b(a^2-b^2)}=\frac{b^2(a-b)}{b(a^2-b^2)}=\frac{b}{a+b}.$$

20. $\dfrac{(n+3)-1-(n+2)}{(n+1)(n+2)(n+3)}=0.$

21. $\dfrac{(a^2-bc)(b+c)+(b^2-ac)(a+c)+(c^2-ab)(a+b)}{(a+b)(a+c)(b+c)}$

$$=\frac{a^2b+a^2c-b^2c-bc^2+ab^2-a^2c+b^2c-ac^2+ac^2-a^2b+bc^2-ab^2}{(a+b)(a+c)(b+c)}$$

$=0.$

LII.

1. $\dfrac{x}{x-y}+\dfrac{x-y}{y-x}=\dfrac{x}{x-y}+\dfrac{-x+y}{x-y}=\dfrac{y}{x-y}.$

2. $\dfrac{(3+2x)(2+x)}{4-x^2}-\dfrac{(2-3x)(2-x)}{4-x^2}+\dfrac{x^2-16x}{4-x^2}$

$=\dfrac{6+7x+2x^2-(4-8x+3x^2)+x^2-16x}{4-x^2}=\dfrac{2-x}{4-x^2}=\dfrac{1}{2+x}.$

3. $\dfrac{x}{x+1}-\dfrac{-x}{x-1}+\dfrac{x^2}{x^2-1}=\dfrac{x^2-x-(-x^2-x)+x^2}{x^2-1}=\dfrac{3x^2}{x^2-1}.$

4. $\dfrac{1}{6(y+1)}-\dfrac{1}{2(y-1)}+\dfrac{4}{3(1-y^2)}=\dfrac{y-1-3(y+1)}{6(y^2-1)}+\dfrac{-4}{3(y^2-1)}$

$=\dfrac{y-1-3y-3-8}{6(y^2-1)}=\dfrac{-2y-12}{6(y^2-1)}=\dfrac{2y+12}{6(1-y^2)}=\dfrac{y+6}{3(1-y^2)}.$

5. $\dfrac{1}{(m-2)(m-3)}+\dfrac{-2}{(m-1)(m-3)}+\dfrac{1}{(m-1)(m-2)}$

$=\dfrac{m-1-2(m-2)+m-3}{(m-1)(m-2)(m-3)}=0.$

6. $\dfrac{1}{(a-b)(x+b)}+\dfrac{-1}{(a-b)(x+a)}=\dfrac{x+a-(x+b)}{(a-b)(x+a)(x+b)}$

$=\dfrac{a-b}{(a-b)(x+a)(x+b)}=\dfrac{1}{(x+a)(x+b)}.$

7. $\dfrac{a^2+b^2}{(a+b)(a-b)}-\dfrac{2ab^2}{(a-b)(a^2+ab+b^2)}+\dfrac{2a^2b}{(a+b)(a^2-ab+b^2)}$

$=\dfrac{(a^2+b^2)(a^2+ab+b^2)-2ab^2(a+b)}{(a+b)(a-b)(a^2+ab+b^2)}+\dfrac{2a^2b}{(a+b)(a^2-ab+b^2)}$

$=\dfrac{a^4+a^2b^2+a^3b+ab^3+a^2b^2+b^4-2a^2b^2-2ab^3}{(a+b)(a-b)(a^2+ab+b^2)}+\dfrac{2a^2b}{(a+b)(a^2-ab+b^2)}$

$=\dfrac{a^4+a^3b-ab^3+b^4}{(a+b)(a^3-b^3)}+\dfrac{2a^2b}{(a+b)(a^2-ab+b^2)}$

$$=\frac{(a^4+a^3b-ab^3+b^4)(a^2-ab+b^2)+2a^2b(a^3-b^3)}{(a+b)(a^3-b^3)(a^2-ab+b^2)}$$

$$=\frac{a^6+2a^2b^4-2ab^5+b^6+2a^5b-2a^2b^4}{(a^3-b^3)(a^3+b^3)}=\frac{a^6+2a^5b-2ab^5+b^6}{a^6-b^6}.$$

8. $$\frac{1}{4(1+x)}-\frac{-1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1-x-(-1-x)}{4(1-x^2)}+\frac{1}{2(1+x^2)}$$

$$=\frac{2}{4(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1}{2(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1+x^2+1-x^2}{2(1-x^4)}$$

$$=\frac{2}{2(1-x^4)}=\frac{1}{1-x^4}.$$

9. $$\frac{1}{(x-y)(y-z)}+\frac{-1}{(x-y)(x-z)}+\frac{1}{(x-z)(y-z)}$$

$$=\frac{(x-z)+(-y+z)+(x-y)}{(x-y)(y-z)(x-z)}=\frac{2(x-y)}{(x-y)(y-z)(x-z)}=\frac{2}{(x-z)(y-z)}$$

10. $$\frac{1}{a(a-b)(a-c)}+\frac{-1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)}$$

$$=\frac{bc(b-c)+ac(-a+c)+ab(a-b)}{abc(a-b)(a-c)(b-c)}$$

$$=\frac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}=\frac{1}{abc}.$$

LIII.

1. $$\frac{1}{(x+4)(x+5)}+\frac{1}{(x+7)(x+5)}=\frac{x+7+x+4}{(x+4)(x+5)(x+7)}$$

$$=\frac{2x+11}{(x+4)(x+5)(x+7)}.$$

2. $$\frac{1}{(x-7)(x-6)}+\frac{1}{(x-9)(x-6)}=\frac{x-9+x-7}{(x-7)(x-6)(x-9)}$$

$$=\frac{2(x-8)}{(x-6)(x-7)(x-9)}.$$

3. $\dfrac{1}{(x+11)(x-4)}+\dfrac{1}{(x+11)(x-13)}=\dfrac{x-13+x-4}{(x-4)(x+11)(x-13)}$

$=\dfrac{2x-17}{(x-4)(x+11)(x-13)}.$

4. $\dfrac{1}{(x+1)(x+2)}+\dfrac{2x}{(x+1)(x+3)}+\dfrac{1}{(x+2)(x+3)}$

$=\dfrac{x+3+2x(x+2)+x+1}{(x+1)(x+2)(x+3)}=\dfrac{2x^2+6x+4}{(x+1)(x+2)(x+3)}$

$=\dfrac{2(x^2+3x+2)}{(x^2+3x+2)(x+3)}=\dfrac{2}{x+3}.$

5. $\dfrac{m^2+mn+2mn}{n(m+n)}-\dfrac{2mn}{(m+n)^2}=\dfrac{(m^2+3mn)(m+n)-2mn^2}{n(m+n)^2}$

$=\dfrac{m^3+3m^2n+m^2n+3mn^2-2mn^2}{n(m+n)^2}=\dfrac{m^3+4m^2n+mn^2}{n(m+n)^2}.$

6. $\dfrac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)}-\dfrac{2}{1+x^2+x^4}$

$=\dfrac{1+x^3+1-x^3}{1+x^2+x^4}-\dfrac{2}{1+x^2+x^4}=0.$

7. $\dfrac{5+5x-6+6x}{3(1-x^2)}+\dfrac{7x}{3(1+x^2)}-\dfrac{-7x}{3(1-x^2)}=\dfrac{11x-1+7x}{3(1-x^2)}+\dfrac{7x}{3(1+x^2)}$

$=\dfrac{(18x-1)(1+x^2)+7x(1-x^2)}{3(1-x^4)}=\dfrac{18x+18x^3-1-x^2+7x-7x^3}{3(1-x^4)}$

$=\dfrac{11x^3-x^2+25x-1}{3(1-x^4)}.$

8. $\dfrac{(3-x)+2(x-1)}{8(x-1)(3-x)}+\dfrac{1}{8(x-5)}+\dfrac{1}{(1-x)(x-3)(x-5)}$

$=\dfrac{(x+1)(x-5)+(x-1)(3-x)}{8(x-1)(3-x)(x-5)}+\dfrac{1}{(x-1)(3-x)(x-5)}$

$$=\frac{x^2-4x-5+4x-x^2-3+8}{8(x-1)(3-x)(x-5)}=0.$$

9. $\dfrac{(1-x+x^2-x^3)(1+x)+x^4}{1+x}=\dfrac{1-x^4+x^4}{1+x}=\dfrac{1}{1+x}.$

LIV.

1. Multiply by 2 ; $x=16$.

2. Multiply by 4 ; $3x=36$, $x=12$.

3. Multiply by 15 ; $5x+3x=120$; $8x=120$, $x=15$.

4. Multiply by 28 ; $7x-4x=84$; $3x=84$, $x=28$.

5. $-\dfrac{4x}{9}=-28$; $\dfrac{4x}{9}=28$, $\dfrac{x}{9}=7$, $x=63$.

6. Multiply by 15 ; $10x=528-12x$; $22x=528$; $x=24$.

7. Multiply by 12 ; $8x+48=7x+108$; $x=60$.

8. $\dfrac{2x}{3}+6=\dfrac{4x}{5}$; $10x+90=12x$; $x=45$.

9. $\dfrac{3x}{4}+3=\dfrac{5x}{6}$; $9x+36=10x$; $x=36$.

10. $\dfrac{7x}{8}=\dfrac{9x}{10}-3$; $35x=36x-120$; $x=120$.

11. $\dfrac{5x}{9}+\dfrac{7x}{12}=82$; $20x+21x=82\times 36$; $x=72$.

12. $\dfrac{x}{6}+\dfrac{x}{8}=28$; $4x+3x=28\times 24$; $x=96$.

13. $\dfrac{5x}{8}-\dfrac{3x}{4}=-8$; $5x-6x=-64$; $x=64$.

14. $9x+360-10x=348$; $-x=-12$; $x=12$.

15. $3x-44=2x-16$; $x=28$.

16. $6x+4x+3x=13$; $13x=13$; $x=1$.

17. Multiply by 70; $14x+28+10x-10=35x-70$; $-11x=-88$; $x=8$.

18. Multiply by 12; $6x+4x=117-3x$; $13x=117$; $x=9$.

19. Multiply by 140; $35x+315+40x=84x-168+420$; $-9x=-63$; $x=7$.

20. Multiply by 105; $357-63x=1015-385x+140x+70$; $182x=728$; $x=4$.

21. Multiply by 7; $2x-10=0$; $2x=10$; $x=5$.

22. Multiply by 329; $141x+188+28x-357=0$; $169x=169$; $x=1$.

23. Multiply by x; $3-3x=1-x$; $-2x=-2$; $x=1$.

24. Multiply by x; $12+x-5x=6$; $-4x=-6$; $x=\frac{3}{2}$.

25. Multiply by 20; $5x+2x+x=800$; $8x=800$; $x=100$.

26. Multiply by 8; $18x+12-4x=29x-348$; $-15x=-360$; $x=24$.

27. Multiply by $100x$; $275x-300=100-325x$; $600x=400$; $x=\frac{2}{3}$.

28. Multiply by 90; $225+540-30x=100x+30+27-18x+36$; $-112x=-672$; $x=6$.

29. Multiply by 12; $4x+3x-10x-144=20x-696$; $-23x=-552$; $x=24$.

30. Multiply by 60; $42x+12-720-45x=36x+156-255x$; $216x=864$; $x=4$.

LV.

1. Multiply by 2; $10x - x - 2 = 142$; $9x = 144$; $x = 16$.

2. Multiply by 3; $3x - 3 + x = 17$; $4x = 20$; $x = 5$.

3. Multiply by 4; $5 - 2x + 8 = 4x - 12x + 16$; $6x = 3$; $x = \frac{1}{2}$.

4. Multiply by 4; $10x - 5x = 9 - 6 + 2x$; $3x = 3$; $x = 1$.

5. Multiply by 30; $60x - 25x + 20 = 210 - 6 + 12x$; $23x = 184$; $x = 8$.

6. Multiply by 36; $18x + 36 = 56 - 27 - 45x$; $63x = -7$; $x = -\frac{1}{9}$.

7. Multiply by 48; $30x + 18 - 48 + 64x + 24x = 744 - 72 + 40x$; $78x = 702$; $x = 9$.

8. Multiply by 385; $55x + 275 - 77x + 154 = 35x + 315$; $-57x = -114$; $x = 2$.

9. Multiply by 105; $35x + 35 - 15x + 60 = 21x + 84$; $-x = -11$; $x = 11$.

10. Multiply by 24; $24x - 72 - 3x - 6 = 8x$; $13x = 78$; $x = 6$.

11. Multiply by 84; $12x + 60 = 21x + 42 - 28x + 56$; $19x = 38$; $x = 2$.

12. Multiply by 33; $11x - 3x + 3 = 33x - 297$; $-25x = -300$; $x = 12$.

13. Multiply by 70; $14x + 28 = 35x - 70 - 10x + 10$; $-11x = -88$; $x = 8$.

14. Multiply by 140; $35x + 315 - 84x + 168 = 420 - 40x$; $-9x = -63$; $x = 7$.

15. Multiply by 78; $39x + 39 - 26x + 78 = 6x + 180$; $7x = 63$; $x = 9$.

16. Multiply by 35; $10x - 7x - 21 = 105x - 735$; $-102x = -714$; $x = 7$.

17. Multiply by 154 ; $44x+154-126x+112=77x-847$;
$-159x=-1113$; $x=7$.

18. Multiply by 572 ; $1001x-4433-176-330x=182x-208$;
$489x=4401$; $x=9$.

19. Multiply by 273 ; $728x-1365-429x+39=147x+42$;
$152x=1368$; $x=9$.

20. Multiply by 56 ; $49x+63-24x-8=126x-182-996+36x$;
$-137x=-1233$; $x=9$.

21. Multiply by 280 ; $28x+2800x=140x+56x+7x-400+40x+26250$; $2585x=25850$; $x=10$.

LVI.

1. $(a+b)x=c$; $x=\dfrac{c}{a+b}$.

2. $5bx-cx=3c-2a$; $(5b-c)\,x=3c-2a$, etc.

3. $ax+fx=a^2b-bc+d$; $(a+f)\,x=a^2b-bc+d$, etc.

4. $ax-5x=bc-dm$; $(a-5)x=bc-dm$, etc.

5. $-a^2x-ax=-a^2b-abc$; $ax+x=ab+bc$; $(a+1)\,x=b\,(a+c)$, etc.

6. $3acx-12cdx=abc+6bcd$; $(3a-12d)\,x=ab+6bd$, etc.

7. $3ackx-kx+ackx=-k^2-3k-k^2+3abk$;
$3acx-x+acx=-k-3-k+3ab$; $(4ac-1)x=3ab-2k-3$:
$x=\dfrac{3ab-2k-3}{4ac-1}$.

8. $abcx-cmx+ac^2x=ac^2+abc-mc$;
$abx-mx+acx=ac+ab-m$; $x=1$.

9. $(a+b)^2-x^2=ab+(b-a)x-x^2-ab$; $x=\dfrac{(a+b)^2}{b-a}$.

10. $a^2-x^2=2a^2+2ax-x^2$; $-2ax=a^2$; $x=-\dfrac{a}{2}$.

11. $a^4+2a^2x+x^2=x^2+4a^2+a^4$; $2a^2x=4a^2$; $x=2$.

12. $a^4-x^2=a^4+2ax-x^2$; $-2ax=0$; $x=0$.

13. $ax-b+ac=x+ac$; $ax-x=b$; $x=\dfrac{b}{a-1}$.

14. $2ax-3a+bx=1$; $2ax+bx=3a+1$; $x=\dfrac{3a+1}{2a+b}$.

15. $18a-4ax+2b=3x$; $3x+4ax=18a+2b$; $x=\dfrac{18a+2b}{4a+3}$.

16. $ax^2-bx-1=ax^2-a$; $-bx=1-a$; $x=\dfrac{a-1}{b}$.

17. $mp^2x+mx^3=mpqx^2+mx^3$; $mp^2x=mpqx^2$; $p=qx$; $x=\dfrac{p}{q}$.

18. $dx-abd=ac-adx$; $dx+adx=abd+ac$; $x=\dfrac{abd+ac}{ad+d}$.

19. $x^2-a-ax+x^2=2x^2-ab$; $-ax=a-ab$; $x=b-1$

20. $3bx-abc+cx^2=4bx-abc$; $cx^2=bx$; $x=\dfrac{b}{c}$.

21. $a^3b+a^2x-b^3+bx=b^2x-b^3-a^3b+a^2x$; $bx-b^2x=-2a^3b$;
$x=\dfrac{2a^3}{b-1}$.

22. $6ax-4b-3ax+3a=6ax-4b$; $-3ax=-3a$; $x=1$.

23. $abm^2-b^2m-amx+bx=0$; $(b-am)x=bm(b-am)$; $x=bm$.

24. $2a^3b^4-b^3x+3a^3bc=3a^2cx(a+b)-ab^4+2a^2b^3x$;
$-b^3x-3a^3cx-3a^2bcx-2a^2b^3x=-3a^3bc-2a^3b^4-ab^4$;
$x=\dfrac{3a^3bc+2a^3b^4+ab^4}{b^3+3a^3c+3a^2bc+2a^2b^3}$.

25. $acx^2+abc-ac^2x+abx-acx^2=0$; $ac^2x-abx=abc$; $x=\dfrac{bc}{c^2-b}$.

26. $ad^2+ax^2=acdx+ax^2$; $x=\dfrac{d}{c}$.

27. $ab=(bc+d)\,x+1$; $x=\dfrac{ab-1}{bc+d}$.

28. $3ac+cx=3a^2+ax+am-mx$; $cx-ax+mx=am-3ac+3a^2$;

$$x=\frac{am-3ac+3a^2}{c-a+m}.$$

29. $ab+ax+bx+x^2-ab-ac=\dfrac{a^2c}{b}+x^2$; $ax+bx=ac+\dfrac{a^2c}{b}$;

$$(a+b)x=\frac{(ab+a^2)\,c}{b}\ ;\ x=\frac{ac}{b}.$$

30. $a^2ce-a^2dx-2abdx-b^2dx-abdx=a^2de-3abdx$;

$$-a^2dx-b^2dx=a^2de-a^2ce\ ;\quad x=\frac{a^2e\,(c-d)}{(a^2+b^2)\,d}.$$

LVII.

1. $(3x+7)(4x+3)=(3x+5)(4x+5)$;
$12x^2+37x+21=12x^2+35x+25$; $2x=4$; $x=2$.

2. $(x+6)(2x-5)=x(2x+5)$; $2x^2+7x-30=2x^2+5x$; $2x=30$; $x=15$.

3. $(2x+7)(2x-1)=(4x-1)(x+2)$; $4x^2+12x-7=4x^2+7x-2$;
$5x=5$; $x=1$.

4. $(5x-1)(2x-3)=(5x-3)(2x+3)$; $10x^2-17x+3=10x^2+9x-9$;

$$-26x=-12\ ;\ x=\frac{6}{13}.$$

5. $4x-3+2(3x-2)=0$; $4x-3+6x-4=0$; $10x=7$; $x=\dfrac{7}{10}$.

6. $2(1-2x)-5(1-5x)=0$; $2-4x-5+25x=0$; $21x=3$; $x=\frac{1}{7}$.

7. $\frac{x+1+x-1}{x^2-1}=\frac{3}{x^2-1}$; $2x=3$; $x=\frac{3}{2}$.

8. $\frac{7x-29}{5x-12}=\frac{8x+19}{18}-\frac{4x+3}{9}$; $\frac{7x-29}{5x-12}=\frac{8x+19-8x-6}{18}$;

$18(7x-29)=13(5x-12)$; $126x-522=65x-156$; $61x=366$;

$x=6$.

9. $\frac{x}{3}-\frac{2}{3}=\frac{x^2-5x}{3x-7}$; $(x-2)(3x-7)=3(x^2-5x)$;

$3x^2-13x+14=3x^2-15x$; $2x=-14$; $x=-7$.

10. $\frac{(3x+2)(x+2)+(2x-4)(x-1)}{x^2+x-2}=5$;

$3x^2+8x+4+2x^2-6x+4=5x^2+5x-10$; $-3x=-18$; $x=6$.

11. Multiply by 210; $35(x+3)-30(11-x)=84(x-4)-10(x-3)$; $35x+105-330+30x=84x-336-10x+30$; $-9x=-81$; $x=9$.

12. $(x+1)(2x+2)=2(x-3)(x+6)$; $2x^2+4x+2=2x^2+6x-36$; $-2x=-38$; $x=19$.

13. $\frac{3x^2(2x+3)+2x+1}{6x^2+3x}=x+1$; $6x^3+9x^2+2x+1=(6x^2+3x)(x+1)$;

$6x^3+9x^2+2x+1=6x^3+9x^2+3x$; $-x=-1$; $x=1$.

14. $\frac{3(x-1)-(x+1)(x+1)}{x^2-1}=\frac{-x^2}{x^2-1}$; $3x-3-x^2-2x-1=-x^2$; $x=4$.

15. $\frac{2(1+x)+8(1-x)}{1-x^2}=\frac{45}{1-x^2}$; $2+2x+8-8x=45$; $-6x=35$;

$x=-\frac{35}{6}$.

16. $\frac{4}{x-8}+\frac{3}{2(x-8)}-\frac{29}{24}=\frac{2}{3(x-8)}$; multiply by $24(x-8)$;

$96+36-29x+232=16$; $-29x=-348$; $x=12$.

17. $x^4-4x^3+20x-24=x^4-4x^3+16x-16$; $4x=8$; $x=2$.

18. $2x^4+2x^3-23x^2+31x=2x^4+2x^3-23x^2+7x+12$; $24x=12$; $x=\frac{1}{2}$.

19. Multiply by $16x$; $4x^2-16x=4x^2-3x-\frac{13}{8}$; $-13x=-\frac{13}{8}$; $x=\frac{1}{8}$

20. $5-x\left(\frac{7x-4}{2x}\right)=\frac{x}{2}-\frac{8x-4}{4}$; $5-\frac{7x-4}{2}=\frac{x}{2}-\frac{4x-2}{2}$;

$10-7x+4=x-4x+2$; $-4x=-12$; $x=3$.

LVIII.

1. $\frac{5x}{10}-2=\frac{25x}{100}+\frac{2x}{10}-1$; $50x-200=25x+20x-100$; $x=20$.

2. $\frac{325x}{100}-\frac{51}{10}+x-\frac{75x}{100}=\frac{39}{10}+\frac{5x}{10}$; $325x-510+100x-75x=390+50x$

$300x=900$; $x=3$.

3. $\frac{125x}{1000}+\frac{x}{100}=13-\frac{2x}{10}+\frac{4}{10}$; $125x+10x=13000-200x+400$;

$335x=13400$; $x=40$.

4. $\frac{3x}{10}+\frac{1305x}{1000}+\frac{5x}{10}=\frac{2295}{100}-\frac{195x}{1000}$;

$300x+1305x+500x=22950-195x$; $2300x=22950$; $x=\frac{459}{46}$.

5. $\frac{2x}{10}-\frac{x}{100}+\frac{5x}{1000}=\frac{117}{10}$; $200x-10x+5x=11700$;

$195x=11700$; $x=60$.

6. $\frac{24x}{10}-\frac{36x-5}{50}=\frac{8x}{10}+\frac{89}{10}$; $120x-36x+5=40x+445$; $44x=440$;

$x=10$.

7. $\frac{24x}{10}-\frac{1075}{100}=\frac{25x}{100}$; $240x-1075=25x$; $215x=1075$; $x=5$.

8. $\frac{5x}{10}+2-\frac{75x}{100}=\frac{4x}{10}-11$; $50x+200-75x=40x-1100$;

$-65x=-1300$; $x=20$.

9. $\frac{405}{900x}=\cdot150$; $\frac{45}{100x}=\cdot15$; $45=15x$; $x=3$.

10. $\frac{25x}{10}-\frac{2+x}{7}\times\frac{-7}{4}=\frac{5}{10}-\frac{5x+3}{8}$; $\frac{5x}{2}+\frac{2+x}{4}=\frac{1}{2}-\frac{5x+3}{8}$;

$20x+4+2x=4-5x-3$; $27x=-3$; $x=-\frac{1}{9}$

11. $\frac{85}{20}-\frac{2}{10x}=\frac{17}{4}-\frac{10-x}{10x}$; $85x-4=85x-20+2x$; $-2x=-16$; $x=8$.

12. $\frac{48x}{600}-\frac{30-40x}{2}=1993$; $8x-1500+2000x=199300$;

$2008x=200800$; $x=100$.

13. $\frac{20-30x}{15}+\frac{500x}{125}-\frac{2x-3}{9}=\frac{10x-20}{18}+\frac{25}{9}$;

$\frac{4-6x}{3}+4x-\frac{2x-3}{9}=\frac{5x-10}{9}+\frac{25}{9}$;

$12-18x+36x-2x+3=5x-10+25$; $11x=0$; $x=0$.

14. $\frac{2408}{100x}+\frac{1}{x}\times\frac{4}{100}\times\frac{10x+9}{10}=\frac{2412}{10}$; $\frac{2408}{100x}+\frac{40x+36}{1000x}=\frac{2412}{10}$;

$24080+40x+36=241200x$; $24116=241160x$; $x=\cdot1$.

15. $\frac{5x}{10}+\frac{45x-75}{60}=\frac{12}{2}-\frac{3x-6}{9}$; $\frac{x}{2}+\frac{3x-5}{4}=6-\frac{x-2}{3}$;

$6x+9x-15=72-4x+8$; $19x=95$: $x=5$.

16. $\frac{1}{2}-\frac{35x}{10x-20}-\frac{24-3x}{8}=\frac{375x}{1000}$; $500-\frac{3500x}{x-2}-3000+375x=375x$;

$-2500=\frac{3500x}{x-2}$; $-25x+50=35x$; $50=60x$; $x=\frac{5}{6}$.

17. $\frac{15x}{100}+\frac{135x-225}{600}=\frac{36}{20}-\frac{9x-18}{90}$; $\frac{3x}{20}+\frac{45x-75}{200}=\frac{18}{10}-\frac{x-2}{10}$;

$30x+45x-75=360-20x+40$; $95x=475$; $x=5$.

LIX.

1. Let x be the number; then $\frac{x}{2}+\frac{x}{4}+\frac{x}{5}=95$, etc.

2. Let x be the number; then $\frac{x}{12}+\frac{x}{20}+\frac{x}{40}=38$, etc.

3. Let x be the number; then $\frac{x}{4}-\frac{x}{5}=4$, etc.

4. Let x be the number; then $\frac{x}{25}-\frac{x}{35}=8$, etc.

5. Let x be one part; then $60-x$ is the other; and $\frac{x}{7}=\frac{60-x}{8}$, etc.

6. Let x be one part; then $50-x$ is the other;

   and $\frac{x}{4}+\frac{5(50-x)}{6}=40$, etc.

7. Let x be the greater; then $100-x$ is the less; and

   $\frac{x}{4}-\frac{100-x}{3}=11$, etc.

8. Let x be the number; then $x-\left(\frac{x}{3}+\frac{x}{10}+\frac{x}{12}\right)=58$, etc.

9. Let x be the number; then $33-\frac{x}{4}-\frac{x}{5}-\frac{x}{10}=0$, etc.

10. Let x be the number; then $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=\frac{x}{2}+112$, etc.

11. Let x be the number; then $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{12}+30=2x$, etc.

12. Let x be the greater; then $x-8$ is the less; and $\frac{x}{x-8}=3$, etc.

13. Let x be the property in pounds; then $\frac{x}{7}=x-1626$, etc.

14. Let x be the greater; then $x-504$ is the less; and $\frac{x}{x-504}=15$, etc.

15. Let x be the greater; then $5760-x$ is the less; and

$$x-(5760-x)=\frac{x}{3} \text{ etc.}$$

16. Let x be the number; then $x+\frac{x}{2}-60=65-x$, etc.

17. Let x be the greater; then $x-20$ is the less; and $\frac{x}{7}=\frac{x-20}{3}$, etc.

18. Let x be the less; then $31207-x$ is the greater; and

$$\frac{31207-x}{x}=15+\frac{1335}{x}, \text{ etc.}$$

19. Let x be the age of the younger; then $27-x$ is the age of the elder; and $\frac{27-x}{x}=\frac{7}{2}$, etc.

20. Let x be the greater; then $237-x$ is the less; and $237-x=\frac{4x}{5}$, etc.

21. Let x be A's share; then $\frac{2x}{7}$ is B's share; and $x+\frac{2x}{7}=1800$, etc.

22. Let x be one part; then $46-x$ is the other; and $\frac{x}{7}+\frac{46-x}{3}=10$, etc.

23. Let x be one part; then $a-x$ is the other; and $\frac{x}{m}+\frac{a-x}{n}=b$, etc.

24. Let x be one number; then $a-x$ is the other; and
$x-(a-x)=b$, etc.

25. Let x be the number; then $\frac{4x}{3}=24$, etc.

26. Let x be B's share; then $\frac{5x}{11}$ is A's share; and $x+\frac{5x}{11}$ is C's share;

then $x+\frac{5x}{11}+x+\frac{5x}{11}=864$, etc.

27. Let x be the property; then $\frac{x}{2}+\frac{x}{6}+\frac{x}{6}+\frac{x}{12}+600=x$, etc.

28. Let x be the second; then $70-x$ is the first; and
$\frac{70-x}{x}=2+\frac{1}{x}$, etc.

29. Let x be the first; then $x+25$ is the second; and
$\frac{x+25}{x}=4+\frac{4}{x}$.

30. Let x be the greater; then $208-x$ is the less; and
$\frac{x}{4}+\frac{208-x}{3}=4\{x-(208-x)\}-4$, etc.

31. Let x be the number of days; then $\frac{2x}{3}-\frac{x}{2}=13$, etc.

32. Let x be the number of gallons; then $\frac{4x}{5}-10=\frac{2x}{3}$.

33. In 25 years the sum of their ages will be increased by 50 years; hence the sum of their ages at the present time is 50 years. Let x be the age of the father; then $50-x$ is the age of the

son; and the difference of their ages is $x-(50-x)$, or, $2x-50$. In 20 years the age of the father will be $x+20$, and the age of the son $70-x$. Hence the sum in 20 years will be $x+20+(70-x)$, or, 90 years. Therefore, by the question, $2x-50=30$, or, $x=40$.

34. Let x be the number of years; then $70-x=\frac{10}{3}(35-x)$, etc.

35. Let x be the number of years; then $72-x=5(48-x)$, etc.

36. Let x be the number of days; then $\frac{1}{2}+\frac{1}{3}=\frac{1}{x}$, etc.

37. Let x be the number of days; then $\frac{1}{50}+\frac{1}{60}+\frac{1}{75}=\frac{1}{x}$, etc.

38. Let x be the number of days; then $\frac{1}{12}+\frac{1}{15}+\frac{1}{20}$ represents the part done daily by *two* men of A's strength, *two* of B's, and *two* of C's, working together. Hence A, B, and C together do $\frac{1}{2}$ of $\left(\frac{1}{12}+\frac{1}{15}+\frac{1}{20}\right)$ daily.

$$\therefore \frac{1}{x}=\frac{1}{2} \text{ of } \frac{5+4+3}{60}, \text{ or, } x=\frac{120}{12}=10.$$

39. Let x be the number of hours; then $\frac{1}{x}$ is the part A does hourly;

A and B do $\frac{1}{4}$ hourly; A and C do $\frac{5}{18}$ hourly;

$\therefore$ B does $\frac{1}{4}-\frac{1}{x}$, and C does $\frac{5}{18}-\frac{1}{x}$ hourly.

$\therefore$ B and C do $\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$ hourly.

Hence $\frac{7}{36}=\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$, etc.

40. Let x be the number of days; then $\frac{2}{5}+\frac{3}{10}+\frac{4}{15}=\frac{1}{x}$, etc.

41. Let x be the number of days; then B will do the whole in $3x$ days.

$\therefore$ B does $\frac{1}{3x}$ daily. Now A does $\frac{3}{50}$ daily, and in 3 days he does $\frac{9}{50}$, while B in 3 days does $\frac{1}{x}$.

Hence $\frac{9}{50}+\frac{1}{x}=\frac{2}{5}$, etc.

42. Let x be the number of hours; then $\frac{1}{3}+\frac{1}{4}=\frac{1}{x}$, etc.

43. Let x be the number of hours; then $\frac{3}{4}+\frac{3}{10}+\frac{1}{5}=\frac{1}{x}$, etc.

44. Let x be the number of minutes; then $\frac{1}{40}-\frac{1}{60}=\frac{1}{x}$, etc.

45. Let x be the number of minutes; then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$, etc.

46. The first pipe lets in 12 gall. in $\frac{13}{4}$ min., or, $\frac{48}{13}$ gall. in a min.

The second „ $\frac{46}{3}$ gall. in $\frac{5}{2}$ min., or, $\frac{92}{15}$ gall. in a min.

The third „ 17 gall. in 3 min., or, $\frac{17}{3}$ gall. in a min.

$\therefore$ All three together let in $\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)$ gall. in a min.

Let x be the number of minutes in which they fill the vessel;

then $x\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)=755\frac{1}{4}$,

or, $\frac{2160x+3588x+3315x}{585}=\frac{3021}{4}$, or, $\frac{9063x}{585}=\frac{3021}{4}$,

or $\frac{3x}{585}=\frac{1}{4}$; $\therefore x=\frac{585}{12}=48\frac{3}{4}$.

47. Let x be the number of gallons let in per minute by the third pipe; then $x+10$ and $x-4$ are the number of gallons let in by the other pipes; then $15(x+x+10+x-4)=2400$, etc.

48. Let x be the distance from Ely; then $16-x$ is the distance from Cambridge; and $\dfrac{x}{4\frac{4}{9}}=\dfrac{16-x}{\frac{60}{18}}$; $\dfrac{9x}{40}=\dfrac{9(16-x)}{30}$; $3x=64-4x$; $x=9\frac{1}{7}$.

49. Let x be the distance up; then x is the distance down; and $\dfrac{x}{2\frac{1}{3}}+\dfrac{x}{3\frac{1}{2}}=5$; $\dfrac{3x}{7}+\dfrac{2x}{7}=5$; $x=7$; and $\therefore 2x=14$.

50. In 1 hour he walks $\dfrac{a}{b}$ miles; hence in c hours he walks $\dfrac{ac}{b}$ miles.

He walks 1 mile in $\dfrac{b}{a}$ hours; hence he walks d miles in $\dfrac{bd}{a}$ hours.

51. Let x be the number of days; then $286x=244(x+2)$, etc.

52. Let x be the number of hours; then $x\times\dfrac{22\frac{1}{2}}{3}=(x+8)\times\dfrac{31\frac{1}{2}}{5}$; $\dfrac{15x}{2}=\dfrac{63(x+8)}{10}$, etc.

53. Let x be the distance from Cambridge; $60-x$ is the distance from London; then $\dfrac{x}{4}=\dfrac{60-x}{3\frac{3}{4}}$; $\dfrac{x}{4}=\dfrac{240-4x}{15}$, etc.

54. Let x be the number of hours; then $\dfrac{5}{3}\times x=\dfrac{7}{5}(x+8)$; $x=42$. Hence 50 hours after A started.

55. (1.) Let the time be x minutes past 1; then $x=5+\dfrac{x}{12}+30$, etc.

(2.) Let the time be x minutes past 4; then $x=20+\dfrac{x}{12}+30$, etc.

(3.) Let the time be x minutes past 8; then $x+30=40+\dfrac{x}{12}$, etc.

56. (1.) Let the time be x minutes past 2; then $x=10+\frac{x}{12}+15$, etc.

(2.) Let the time be x minutes past 4; then there are two solutions, (α) $x+15=20+\frac{x}{12}$, etc. (β) $x=20+\frac{x}{12}+15$, etc.

(3.) Let the time be x minutes past 7; then there are two solutions, (α) $x+15=35+\frac{x}{12}$, etc. (β) $x=35+\frac{x}{12}+15$, etc.

57. (1.) Let the time be x minutes past 3; then $x=15+\frac{x}{12}$, etc.

(2.) Let the time be x minutes past 6; then $x=30+\frac{x}{12}$, etc.

(3.) Let the time be x minutes past 9; then $x=45+\frac{x}{12}$, etc.

58. Let $2x$ be the number of apples; he pays for them $\frac{4x}{5}$ pence.

He sells x for $\frac{x}{2}$ pence, and x for $\frac{x}{3}$ pence.

Hence $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+1$; $x=30$; $2x=60$.

59. Let x be the number of sovereigns he had at first.

Then he has $\frac{x}{2}-\frac{1}{2}$ left after the first gift;

and he has $\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}$ left after the second gift.

Then $\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}=0$; $\frac{x-1}{4}=\frac{1}{2}$; $x=3$.

60. $\dfrac{\frac{2}{3}+n}{3n+\frac{69}{3}}=\frac{1}{33}$; $\frac{2+3n}{9n+69}=\frac{1}{33}$; $66+99n=9n+69$, etc.

61. Let x be the number of days; then $x-3$ is the number of days that the pursuing army is in motion; and $18x+25=23(x-3)$, etc.

62. Let x be his original income in pounds;

then $x - \frac{x}{40} - \frac{1}{13}\left(x - \frac{x}{40}\right) = 540$, etc.

63. Let x be the original sum; then first remainder is $\frac{x}{2} - 50$, or, $\frac{x-100}{2}$;

second remainder is $\frac{4}{5}$ of $\frac{x-100}{2} - 30$, or, $\frac{2x-200-150}{5}$;

third remainder is $\frac{3}{4}$ of $\frac{2x-350}{5} - 20$, or, $\frac{6x-1050-400}{20}$;

thus $\frac{6x-1450}{20} = 10$; $6x = 1650$; $x = 275$.

64. Let $2x$ be the number of eggs bought; for x I gave $\frac{x}{2}$ pence, and for x I gave $\frac{x}{3}$ pence; and I sold $2x$ eggs for $\frac{4x}{5}$ pence.

Thus $\frac{x}{2} + \frac{x}{3} = \frac{4x}{5} + 1$; $x = 30$; $2x = 60$.

65. Let x be the number of minutes in which C would fill it; then

$\frac{1200}{x}$ = number of gall. let in by C each minute; and therefore

$\frac{1200}{x} + 10$ = number of gall. let in by A and B each minute.

Hence $\frac{1200}{x} + 10 + \frac{1200}{x} = \frac{1200}{24}$; whence $x = 60$.

Thus A takes 90 minutes and C 60 minutes.

Also B lets in $\left(\frac{1200}{24} - \frac{1200}{90} - \frac{1200}{60}\right)$ gall. per minute, or,

$50 - \frac{40}{3} - 20$, or, $\frac{50}{3}$ gall. per minute.

$\therefore$ B takes $1200 \div \frac{50}{3}$ minutes, or, 72 minutes to fill it.

66. Let x be the number of days B would take; then $2x$ is the number A would take.

Then $\frac{1}{x}+\frac{1}{2x}$ is the part A and B drink daily,

$\therefore$ $\frac{3}{4}$ of $\left(\frac{1}{x}+\frac{1}{2x}\right)$ is the part C drinks daily.

Then $\frac{1}{x}+\frac{1}{2x}+\frac{3}{4x}+\frac{3}{8x}=\frac{1}{24}$; $x=63$, etc.

67. Let x be the number of shots each fires.

Then $\frac{7x}{12}$ is the number of A's hits, and $\frac{9x}{12}$ of B's hits.

Thus $\frac{7x}{12}+\frac{9x}{12}=32$; $x=24$.

68. Let x be the number of horses; then $2x$ is the number of oxen, and $100-3x$ is the number of sheep. Reducing all the prices to shillings,

$$440x+250\times 2x+30(100-3x)=4700;\ x=2, \text{ etc.}$$

69. Let x be the whole number of marks.

Then $\frac{x}{5}=$ number of marks obtained by B.

And $\frac{2x}{5}=$number of marks assigned to book work in the paper.

Also $\frac{2x}{5}=$number of marks obtained by A for riders.

$\therefore$ $\frac{2x}{5}+\frac{2x}{5}=160$; $x=200$.

70. Since the man made a mistake of 55 minutes, the distance between the hands was 5 minutes.

Let x be the number of minutes past 2.

Then $x+5=10+\frac{x}{12}$; $x=5\frac{5}{11}$.

71. Let x be the original force; then $\frac{5x}{6}-4000=$remainder after the defeat.

Hence $\frac{5x}{6}-1000$ = number after the reinforcement.

Then $\frac{3}{4}\left(\frac{5x}{6}-1000\right)=18000$; $\frac{5x}{8}-750=18000$; $x=30000$.

72. Let x be the original debt; then $x+\frac{x}{4}-25000000$ = debt after the peace.

Then $\frac{4\frac{1}{2}\times x}{100}=\frac{4\left(x+\frac{x}{4}-25000000\right)}{100}$; $\frac{9x}{2}=5x-100000000$, etc.

73. The influx goes on for 10 days and 15 hours, or, 255 hours.
The consumption extends over 10 days (excluding Sunday) of 15 hours each, or, 150 hours.
Hence if x be the number of gallons that come in each hour,

$$255x+2250=150\times 2x;\ x=50.$$

LX.

1. $a+x+\frac{3a}{x}=\frac{ax+x^2+3a}{x}$.

2. $\frac{a^2+ax}{x^2}-\frac{2x-2a}{x}=\frac{a^2+ax-2x^2+2ax}{x^2}=\frac{a^2+3ax-2x^2}{x^2}$.

3. $\frac{x-y}{x}+\frac{2y}{x-y}=\frac{x^2-2xy+y^2+2xy}{x(x-y)}=\frac{x^2+y^2}{x(x-y)}$.

4. $\frac{4a+4b}{a-b}-\frac{2a^2-2b^2}{a^2+b^2}=\frac{4a^3+4ab^2+4a^2b+4b^3-2a^3+2ab^2+2a^2b-2b^3}{(a-b)(a^2+b^2)}$, etc

LXI.

1. $\frac{5x+4}{35}-\frac{23x}{70}=\frac{10x+8-23x}{70}=\frac{8-13x}{70}$.

2. $\frac{\frac{x^2-y^2}{xy}}{x-y}=\frac{x^2-y^2}{xy(x-y)}=\frac{x+y}{xy}$.

3. $\dfrac{1-x^2}{\dfrac{x+1}{x}}=\dfrac{x(1-x^2)}{1+x}=x(1-x).$

4. $\dfrac{y\left(\dfrac{x+y}{y}\right)}{x\left(\dfrac{x-y}{x}\right)}=\dfrac{x+y}{x-y}.$

5. $\dfrac{5x^2+x^3+1}{x^2}\div\dfrac{2x^2-x^3+1}{x^2}=\dfrac{5x^2+x^3+1}{2x^2-x^3+1}.$

6. $\dfrac{x^3+1}{x^2}\div\dfrac{x+1}{x}=\dfrac{(x+1)(x^2-x+1)x}{x^2(x+1)}=\dfrac{x^2-x+1}{x}.$

7. $\dfrac{a^3-1}{a^2}\div\dfrac{a-1}{a}=\dfrac{(a-1)(a^2+a+1)a}{a^2(a-1)}=\dfrac{a^2+a+1}{a}.$

8. $\dfrac{x^2-ax+x^2+ax}{x^2-a^2}\div\dfrac{2x}{x^2-a^2}=\dfrac{2x^2}{2x}=x.$

9. $\dfrac{2x}{x^2+x^2}=\dfrac{2x}{2x^2}=\dfrac{1}{x}.$

10. $\dfrac{x^2}{1+x}+1-\dfrac{1}{1+x}=\dfrac{x^2+1+x-1}{1+x}=\dfrac{x(1+x)}{1+x}=x.$

11. $\dfrac{x^2-2xy+y^2+x^2+2xy+y^2}{x^2-y^2}\div\dfrac{x^2-2xy+y^2-x^2-2xy-y^2}{x^2-y^2}$

$=\dfrac{2x^2+2y^2}{-4xy}$, etc.

12. $\dfrac{1+x+x^2}{1}\div\dfrac{x^2+x+1}{x^2}=x^2.$

13. $\dfrac{a^2+2ab+b^2+b^2}{b(a+b)}\div\dfrac{a+b}{ab}=\dfrac{(a^2+2ab+2b^2)ab}{b(a+b)(a+b)}$, etc.

14. $\dfrac{2m^2-3m+1}{m}\div\dfrac{2m-1}{m}=\dfrac{2m^2-3m+1}{2m-1}=m-1.$

15. $\dfrac{c+b+a}{abc}\div\dfrac{(a+b+c)(a-b-c)}{ab}=\dfrac{ab(a+b+c)}{abc(a+b+c)(a-b-c)}$, etc.

LXII.

1. $\dfrac{a^4}{2a^4}+\dfrac{3a^3}{2a^4}+\dfrac{2a^2}{2a^4}+\dfrac{5a}{2a^4}=\dfrac{1}{2}+\dfrac{3}{2a}+\dfrac{1}{a^2}+\dfrac{5}{2a^3}.$

2. $\dfrac{a^2bc}{abcd}+\dfrac{ab^2d}{abcd}+\dfrac{abc^2}{abcd}+\dfrac{bcd^2}{abcd}=\dfrac{a}{d}+\dfrac{b}{c}+\dfrac{c}{d}+\dfrac{d}{a}.$

3. $\dfrac{x^3}{x^2y^2}-\dfrac{3x^2y}{x^2y^2}+\dfrac{3xy^2}{x^2y^2}-\dfrac{y^3}{x^2y^2}=\dfrac{x}{y^2}-\dfrac{3}{y}+\dfrac{3}{x}-\dfrac{y}{x^2}.$

4. $\dfrac{9a^3}{108}-\dfrac{12a^2}{108}+\dfrac{6a}{108}-\dfrac{3}{108}=\dfrac{a^3}{12}-\dfrac{a^2}{9}+\dfrac{a}{18}-\dfrac{1}{36}.$

5. $\dfrac{18p^2}{3pqrs}+\dfrac{12q^2}{3pqrs}-\dfrac{36r^2}{3pqrs}+\dfrac{72s^2}{3pqrs}=\dfrac{6p}{qrs}+\dfrac{4q}{prs}-\dfrac{12r}{pqs}+\dfrac{24s}{pqr}.$

6. $\dfrac{10x^3}{1000}-\dfrac{25x^2}{1000}+\dfrac{75x}{1000}-\dfrac{125}{1000}=\dfrac{x^3}{100}-\dfrac{x^2}{40}+\dfrac{3x}{40}-\dfrac{1}{8}.$

LXIII.

1. $1+a)\,2\,(2-2a+2a^2-2a^3+2a^4\ldots$

$$
\begin{array}{l}
\underline{2+2a} \\
\quad -2a \\
\quad \underline{-2a-2a^2} \\
\qquad\quad 2a^2 \\
\qquad\quad \underline{2a^2+2a^3} \\
\qquad\qquad\quad -2a^3 \\
\qquad\qquad\quad \underline{-2a^3-2a^4} \\
\qquad\qquad\qquad\qquad 2a^4 \\
\qquad\qquad\qquad\qquad \underline{2a^4+2a^5} \\
\qquad\qquad\qquad\qquad\quad -2a^5
\end{array}
$$

2. $m+2)\,m\,(1-\dfrac{2}{m}+\dfrac{4}{m^2}-\dfrac{8}{m^3}+\dfrac{16}{m^4}\cdots$

$$\underline{m+2}$$
$$-2$$
$$\underline{-2-\frac{4}{m}}$$
$$\frac{4}{m}$$
$$\underline{\frac{4}{m}+\frac{8}{m^2}}$$
$$-\frac{8}{m^2}$$
$$\underline{-\frac{8}{m^2}-\frac{16}{m^3}}$$
$$\frac{16}{m^3}$$

3. $a+b)\,a-b\,(1-\dfrac{2b}{a}+\dfrac{2b^2}{a^2}-\dfrac{2b^3}{a^3}+\dfrac{2b^4}{a^4}\cdots$

$$\underline{a+b}$$
$$-2b$$
$$\underline{-2b-\frac{2b^2}{a}}$$
$$\frac{2b^2}{a}$$
$$\underline{\frac{2b^2}{a}+\frac{2b^3}{a^2}}$$
$$-\frac{2b^3}{a^2}$$
$$\underline{-\frac{2b^3}{a^2}-\frac{2b^4}{a^3}}$$
$$\frac{2b^4}{a^3}$$

4. $a^2-x^2)\,a^2+x^2\,(1+\dfrac{2x^2}{a^2}+\dfrac{2x^4}{a^4}+\dfrac{2x^6}{a^6}+\dfrac{2x^8}{a^8}\cdots$

$$\underline{a^2-x^2}$$
$$2x^2$$
$$\underline{2x^2-\frac{2x^4}{a^2}}$$
$$\frac{2x^4}{a^2}$$
$$\underline{\frac{2x^4}{a^2}-\frac{2x^6}{a^4}}$$
$$\frac{2x^6}{a^4}$$
$$\underline{\frac{2x^6}{a^4}-\frac{2x^8}{a^6}}$$
$$\frac{2x^8}{a^6}$$

5. $a-x)\,ax\,(x+\dfrac{x^2}{a}+\dfrac{x^3}{a^2}+\dfrac{x^4}{a^3}+\dfrac{x^5}{a^4}\cdots$

$$\underline{ax-x^2}$$
$$x^2$$
$$\underline{x^2-\frac{x^3}{a}}$$
$$\frac{x^3}{a}$$
$$\underline{\frac{x^3}{a}-\frac{x^4}{a^2}}$$
$$\frac{x^4}{a^2}$$
$$\underline{\frac{x^4}{a^2}-\frac{x^5}{a^3}}$$
$$\frac{x^5}{a^3}$$

6. $a+x)\,b\,(\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}\ldots$

$$b+\frac{bx}{a}$$
$$-\frac{bx}{a}$$
$$-\frac{bx}{a}-\frac{bx^2}{a^2}$$
$$\frac{bx^2}{a^2}$$
$$\frac{bx^2}{a^2}-\frac{bx^3}{a^3}$$
$$\frac{bx^3}{a^3}$$
$$\frac{bx^3}{a^3}-\frac{bx^4}{a^4}$$
$$\frac{bx^4}{a^4}$$

7. $1+2x-2x^2)\,1\,(1-2x+6x^2-16x^3+44x^4\ldots$

$$1+2x-2x^2$$
$$-2x+2x^2$$
$$-2x-4x^2+4x^3$$
$$6x^2-4x^3$$
$$6x^2+12x^3-12x^4$$
$$-16x^3+12x^4$$
$$-16x^3-32x^4+32x^5$$
$$-44x^4-32x^5$$

8. $1-x+x^2)\,1+x\,(1+2x+x^2-x^3-2x^4\ldots$

$$1-x+x^2$$
$$2x-x^2$$
$$2x-2x^2+2x^3$$
$$x^2-2x^3$$
$$x^2-x^3+x^4$$
$$-x^3-x^4$$
$$-x^3+x^4-x^5$$
$$-2x^4+x^5$$

9. $1-2b)\,1+b\,(1+3b+6b^2+12b^3+24b^4\ldots$

$$1-2b$$
$$3b$$
$$3b-6b^2$$
$$6b^2$$
$$6b^2-12b^3$$
$$12b^3$$
$$12b^3-24b^4$$
$$24b^4$$

10. $x+b)\,x^3-b^3\,(x^2-bx+b^2-\dfrac{2b^3}{x}+\dfrac{2b^4}{x^2}\cdots$

$$\underline{x^3+bx^2}$$
$$-bx^2-b^3$$
$$\underline{-bx^2-b^2x}$$
$$b^2x-b^3$$
$$\underline{b^2x+b^3}$$
$$-2b^3$$
$$\underline{-2b^3-\frac{2b^4}{x}}$$
$$\frac{2b^4}{x}$$

11. $x-b)\,a^2\,(\dfrac{a^2}{x}+\dfrac{a^2b}{x^2}+\dfrac{a^2b^2}{x^3}+\dfrac{a^2b^3}{x^4}+\dfrac{a^2b^4}{x^5}\cdots$

$$\underline{a^2-\frac{a^2b}{x}}$$
$$\frac{a^2b}{x}$$
$$\underline{\frac{a^2b}{x}-\frac{a^2b^2}{x^2}}$$
$$\frac{a^2b^2}{x^2}$$
$$\underline{\frac{a^2b^2}{x^2}-\frac{a^2b^3}{x^3}}$$
$$\frac{a^2b^3}{x^3}$$
$$\underline{\frac{a^2b^3}{x^3}-\frac{a^2b^4}{x^4}}$$
$$\frac{a^2b^4}{x^4}$$

12. $a^2+2ax+x^2)\,a^2\,(1-\dfrac{2x}{a}+\dfrac{3x^2}{a^2}-\dfrac{4x^3}{a^3}+\dfrac{5x^4}{a^4}\cdots$

$$\underline{a^2+2ax+x^2}$$
$$-2ax-x^2$$
$$\underline{-2ax-4x^2-\frac{2x^3}{a}}$$
$$3x^2+\frac{2x^3}{a}$$
$$\underline{3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}}$$
$$-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$
$$\underline{-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}}$$
$$\frac{5x^4}{a^2}+\frac{4x^5}{a^3}$$

13. Dividend $=$ (Divisor and Quotient) + Remainder

$$=(x-a)(x^2-2ax)+4a^3=x^3-3ax^2+2a^2x+4a^3.$$

14. Dividend $=(m-5)(m^3+5m^2+15m+34)+75$

$$=m^4-10m^2-41m-95.$$

LXIV.

1. $\dfrac{10x^2+15x+6}{30}\times\dfrac{4x+3}{12}=\dfrac{40x^3+90x^2+69x+18}{360}$, etc.

2. $\dfrac{6a^2-5a+10}{30}\times\dfrac{5a-4}{20}=\dfrac{30a^3-49a^2+70a-40}{600}$, etc.

3.
$$\begin{array}{l}
x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}\\
x-\dfrac{1}{x}\\
\hline
x^4+x^2+1+\dfrac{1}{x^2}\\
\quad -x^2-1-\dfrac{1}{x^2}-\dfrac{1}{x^4}\\
\hline
x^4 \qquad\qquad -\dfrac{1}{x^4}
\end{array}$$

4.
$$\begin{array}{l}
x^2-1+\dfrac{1}{x^2}\\
x^2+1+\dfrac{1}{x^2}\\
\hline
x^4-x^2+1\\
\quad +x^2-1+\dfrac{1}{x^2}\\
\qquad +1-\dfrac{1}{x^2}+\dfrac{1}{x^4}\\
\hline
x^4 \quad +1 \qquad +\dfrac{1}{x^4}
\end{array}$$

5. $\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)=\dfrac{1}{a^4}-\dfrac{1}{b^4}.$

6. $\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{ab}-\dfrac{1}{b^2}+\dfrac{1}{bc}+\dfrac{1}{ac}-\dfrac{1}{bc}+\dfrac{1}{c^2}=\dfrac{1}{a^2}+\dfrac{2}{ac}-\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

7. $1+\dfrac{b}{a}+\dfrac{b^2}{a^2}-\dfrac{b}{a}-\dfrac{b^2}{a^2}-\dfrac{b^3}{a^3}+\dfrac{b^2}{a^2}+\dfrac{b^3}{a^3}+\dfrac{b^4}{a^4}=1+\dfrac{b^2}{a^2}+\dfrac{b^4}{a^4}.$

8. $$\begin{array}{l}
1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3 \\
1+\frac{1}{2}x+\frac{1}{4}x^2 \\
\hline
1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3 \\
\quad +\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{16}x^3-\frac{1}{32}x^4 \\
\qquad\quad +\frac{1}{4}x^2-\frac{1}{8}x^3+\frac{1}{32}x^4-\frac{1}{64}x^6 \\
\hline
1+\frac{x^2}{8}\qquad\quad -\frac{x^3}{8}\quad -\frac{x^6}{64}.
\end{array}$$

9. $\dfrac{15+18x-14x^2}{6x^2}\times\dfrac{4-2x-x^2}{2x^2}=\dfrac{60+42x-107x^2+10x^3+14x^4}{12x^4}$, etc.

10. $\dfrac{a^4}{b^4}+1+\dfrac{2a^2}{b^2}-1-\dfrac{b^4}{a^4}-\dfrac{2b^2}{a^2}-\dfrac{2a^2}{b^2}-\dfrac{2b^2}{a^2}-4=\dfrac{a^4}{b^4}-\dfrac{b^4}{a^4}-\dfrac{4b^2}{a^2}-4.$

LXV.

1. $\dfrac{x^4-1}{x^2}\div\dfrac{x^2+1}{x}=\dfrac{x^2-1}{x}$, etc.

2. $\dfrac{a^2b^2-1}{b^2}\div\dfrac{ab-1}{b}=\dfrac{ab+1}{b}$, etc.

3. $\dfrac{m^3n^3+1}{n^3}\div\dfrac{mn+1}{n}=\dfrac{m^2n^2-mn+1}{n^2}$, etc.

4. $\dfrac{c^5d^5-1}{d^5}\div\dfrac{cd-1}{d}=\dfrac{c^4d^4+c^3d^3+c^2d^2+cd+1}{d^4}$, etc.

5. $\dfrac{x^4+2x^2y^2+y^4}{x^2y^2}\div\dfrac{x^2+y^2}{xy}=\dfrac{x^2+y^2}{xy}$, etc.

6. $\dfrac{b^4+a^2b^2+a^4}{a^4b^4}\div\dfrac{b^2-ab+a^2}{a^2b^2}=\dfrac{b^2+ab+a^2}{a^2b^2}$, etc.

7. $\dfrac{x^6-y^6-3x^4y^2+3x^2y^4}{x^3y^3} \div \dfrac{x^2-y^2}{xy} = \dfrac{x^4-2x^2y^2+y^4}{x^2y^2}$, etc.

8. $\dfrac{6x^5-32x^4+77x^3-86x^2-66x+216}{8} \div \dfrac{x^2-2x+6}{2}$

$= \dfrac{6x^3-10x^2+16x+36}{4}$, etc.

9. $\dfrac{a^6+b^6}{a^3b^3} \div \dfrac{a^2+b^2}{ab} = \dfrac{a^4-a^2b^2+b^4}{a^2b^2}$, etc.

10. $\dfrac{b^3c^3+a^3c^3+a^3b^3-3a^2b^2c^2}{a^3b^3c^3} \div \dfrac{bc+ac+ab}{abc}$

$= \dfrac{b^2c^2-abc^2-ab^2c+a^2c^2-a^2bc+a^2b^2}{a^2b^2c^2}$, etc.

LXVI.

1. $\dfrac{x-3}{10} \times \dfrac{50x+7}{100} = \dfrac{50x^2-143x-21}{1000}$, etc.

2. $\left(\dfrac{5x}{100}+7\right) \times \left(\dfrac{2x}{10}-3\right) = \dfrac{5x+700}{100} \times \dfrac{2x-30}{10} = \dfrac{10x^2+1250x-21000}{1000}$, etc.

3. $\dfrac{3x-2y}{10} \times \dfrac{4x+7y}{10} = \dfrac{12x^2+13xy-14y^2}{100}$, etc.

4. $\dfrac{43x+52y}{10} \times \dfrac{4x-6y}{100} = \dfrac{172x^2-50xy-312y^2}{1000}$, etc.

5. $\cdot 000027 - \cdot 001 + \cdot 000343 + \cdot 00063 = \cdot 001 - \cdot 001 = 0$.

6. $\cdot 343 - \cdot 0441 + \cdot 00189 - \cdot 000027 = \cdot 34489 - \cdot 044127 = \cdot 300763$.

LXVII.

1. $a_1x\left(\dfrac{a_1x+a_2x^2+a_3x^3+a_4x^4+\dots}{a_1x}\right) = a\,x\left(1+\dfrac{a_2x}{a_1}+\dfrac{a_3x^2}{a_1}+\dfrac{a_4x^3}{a_1}+\dots\right)$.

2. $xyz\left(\dfrac{xy-xz+yz}{xyz}\right)=xyz\left(\dfrac{1}{z}-\dfrac{1}{y}+\dfrac{1}{x}\right).$

3. $x^2\left(\dfrac{x^2+xy+y^2}{x^2}\right)=x^2\left(1+\dfrac{y}{x}+\dfrac{y^2}{x^2}\right).$

4. $(a+b)\left\{\dfrac{(a+b)^3-c(a+b)^2-d(a+b)+e}{a+b}\right\}$

$$=(a+b)\left\{(a+b)^2-c(a+b)-d+\frac{e}{a+b}\right\}.$$

LXVIII.

1. Let $\dfrac{a}{b}=\lambda$; then $\dfrac{c}{d}=\lambda$; and $a=\lambda b$, $c=\lambda d$. Then

(1.) $\dfrac{a-b}{b}=\dfrac{\lambda b-b}{b}=\lambda-1$

$\dfrac{c-d}{d}=\dfrac{\lambda d-d}{d}=\lambda-1.$

(2.) $\dfrac{a}{a+b}=\dfrac{\lambda b}{\lambda b+b}=\dfrac{\lambda}{\lambda+1}$

$\dfrac{c}{c+d}=\dfrac{\lambda d}{\lambda d+d}=\dfrac{\lambda}{\lambda+1}.$

(3.) $\dfrac{3a}{4a-5b}=\dfrac{3\lambda b}{4\lambda b-5b}=\dfrac{3\lambda}{4\lambda-5}$

$\dfrac{3c}{4c-5d}=\dfrac{3\lambda d}{4\lambda d-5d}=\dfrac{3\lambda}{4\lambda-5}.$

(4.) $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{\lambda^2b^2+b^2}{\lambda^2b^2-b^2}=\dfrac{\lambda^2+1}{\lambda^2-1}$

$\dfrac{c^2+d^2}{c^2-d^2}=\dfrac{\lambda^2d^2+d^2}{\lambda^2d^2-d^2}=\dfrac{\lambda^2+1}{\lambda^2-1}.$

(5.) $\dfrac{8a+b}{4a+7b}=\dfrac{8\lambda b+b}{4\lambda b+7b}=\dfrac{8\lambda+1}{4\lambda+7}$

$\dfrac{8c+d}{4c+7d}=\dfrac{8\lambda d+d}{4\lambda d+7d}=\dfrac{8\lambda+1}{4\lambda+7}.$

(6.) $\dfrac{a^2-b^2}{c^2-d^2}=\dfrac{\lambda^2b^2-b^2}{\lambda^2d^2-d^2}=\dfrac{b^2(\lambda^2-1)}{d^2(\lambda^2-1)}=\dfrac{b^2}{d^2}$

$\dfrac{ab}{cd}=\dfrac{\lambda b\times b}{\lambda d\times d}=\dfrac{b^2}{d^2}$.

(7.) $\dfrac{11a+b}{11c+d}=\dfrac{11\lambda b+b}{11\lambda d+d}=\dfrac{b(11\lambda+1)}{d(11\lambda+1)}=\dfrac{b}{d}$

$\dfrac{13a+b}{13c+d}=\dfrac{13\lambda b+b}{13\lambda d+d}=\dfrac{b(13\lambda+1)}{d(13\lambda+1)}=\dfrac{b}{d}$.

(8.) $\dfrac{a^2-ab+b^2}{a^2+ab+b^2}=\dfrac{\lambda^2b^2-\lambda b^2+b^2}{\lambda^2b^2+\lambda b^2+b^2}=\dfrac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}$

$\dfrac{c^2-cd+d^2}{c^2+cd+d^2}=\dfrac{\lambda^2d^2-\lambda d^2+d^2}{\lambda^2d^2+\lambda d^2+d^2}=\dfrac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}$.

2. Let each of the fractions $=\lambda$

Then $l=a\lambda-b\lambda$; $m=b\lambda-c\lambda$; $n=c\lambda-a\lambda$

And $l+m+n=a\lambda-b\lambda+b\lambda-c\lambda+c\lambda-a\lambda=0$.

3. Let each of the fractions $=\lambda$

Then $a=\lambda b$, $c=\lambda d$, $e=\lambda f$

And $\dfrac{la+mc+ne}{lb+md+nf}=\dfrac{l\lambda b+m\lambda d+n\lambda f}{lb+md+nf}=\lambda=\dfrac{a}{b}$.

4. Let each of the fractions $=\lambda$

Then $a+b=\lambda b$

$b+c=\lambda c$

$c+a=\lambda a$

Adding, $2a+2b+2c=\lambda(a+b+c)$

Hence $\lambda=2$, and $\therefore a+b=2b$, or $a=b$

Similarly we can show that $a=c$, and $b=c$.

5. Let each of the fractions $=\lambda$

Then $a_1=\lambda b_1$; $a_2=\lambda b_2$; $a_3=\lambda b_3$

And $\dfrac{2a_1+3a_2+4a_3}{2b_1+3b_2+4b_3}=\dfrac{2\lambda b_1+3\lambda b_2+4\lambda b_3}{2b_1+3b_2+4b_3}=\lambda=\dfrac{a_1}{b_1}$.

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}$ is less than λ; and $\frac{e}{f}$ is less than λ

$\therefore a=\lambda b$; c is less than λd; e is less than λf

$\therefore a+c+e$ is less than $\lambda b+\lambda d+\lambda f$

$\therefore \frac{a+c+e}{b+d+f}$ is less than λ, that is, less than $\frac{a}{b}$

Next let $\frac{e}{f}=\mu$; then $\frac{a}{b}$ and $\frac{c}{d}$ are each greater than μ

$\therefore e=\mu f$; a is greater than μb; c is greater than μd

$\therefore a+c+e$ is greater than $\mu b+\mu d+\mu f$

$\therefore \frac{a+c+e}{b+d+f}$ is greater than μ, that is than $\frac{e}{f}$.

7. Let each fraction $=\lambda$; then $x_1=\lambda y_1$; and $x_2=\lambda y_2$

Then $\frac{4x_1+5y_1}{7x_1+9y_1}=\frac{4\lambda y_1+5y_1}{7\lambda y_1+9y_1}=\frac{4\lambda+5}{7\lambda+9}$

And $\frac{4x_2+5y_2}{7x_2+9y_2}=\frac{4\lambda y_2+5y_2}{7\lambda y_2+9y_2}=\frac{4\lambda+5}{7\lambda+9}$.

8. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{a^2+ab}{c^2+cd}=\frac{\lambda^2b^2+\lambda b^2}{\lambda^2d^2+\lambda d^2}=\frac{b^2(\lambda^2+\lambda)}{d^2(\lambda^2+\lambda)}=\frac{b^2}{d^2}$

And $\frac{ab-b^2}{cd-d^2}=\frac{\lambda b^2-b^2}{\lambda d^2-d^2}=\frac{b^2(\lambda-1)}{d^2(\lambda-1)}=\frac{b^2}{d^2}$.

9. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{7a+b}{3a+5b}=\frac{7\lambda b+b}{3\lambda b+5b}=\frac{7\lambda+1}{3\lambda+5}$

And $\frac{7c+d}{3c+5d}=\frac{7\lambda d+d}{3\lambda d+5d}=\frac{7\lambda+1}{3\lambda+5}$.

10. Since b is greater than a

bc is greater than ac

$ab+bc$ is greater than $ab+ac$

$\frac{ab+bc}{b(b+c)}$ is greater than $\frac{ab+ac}{b(b+c)}$

$\frac{b(a+c)}{b(b+c)}$ is greater than $\frac{a(b+c)}{b(b+c)}$

$\frac{a+c}{b+c}$ is greater than $\frac{a}{b}$.

11. Since b is less than a, we can show, as in the preceding solution that $\dfrac{a+c}{b+c}$ is less than $\dfrac{a}{b}$.

LXIX.

1. $$3 \times 16 + \frac{2 \times 4 \times \dfrac{1}{4}}{1} - \frac{1}{\dfrac{1}{4}} = 48 + 2 - 4 = 46.$$

2.
$$\begin{array}{l}
7x^2 - 12x + 5)\,14x^3 + 7x^2 - 56x + 35\,(2x \\
\qquad\qquad\quad 14x^3 - 24x^2 + 10x \\
\qquad\qquad\qquad\quad 31x^2 - 66x + 35 \\
\qquad\qquad\qquad\qquad 7 \\
\qquad\qquad\qquad\quad 217x^2 - 462x + 245\,(31 \\
\qquad\qquad\qquad\quad 217x^2 - 372x + 155 \\
\qquad\qquad\qquad\qquad\quad - 90x + 90
\end{array}$$

Hence H. C. F. is $x-1$, and dividing both terms of the first fraction by it, we get $\dfrac{2x^2+3x-5}{7x-5}$

$$\begin{array}{l}
a^2 + 4a - 45)\,a^3 - 39a + 70\,(a - 4 \\
\qquad\qquad\quad a^3 + 4a^2 - 45a \\
\qquad\qquad\quad - 4a^2 + 6a + 70 \\
\qquad\qquad\quad - 4a^2 - 16a + 180 \\
\qquad\qquad\qquad\quad 22a - 110
\end{array}$$

Hence we find $a-5$ to be the H. C. F., and dividing both terms of the fraction by it, we get $\dfrac{a^2+5a-14}{a+9}$.

3. $$\frac{a^2 + 2ap + p^2 - a^2 + 2ap - p^2}{a^2 - p^2} \div \frac{a^2 + 2ap + p^2 + a^2 - 2ap + p^2}{a^2 - p^2}$$

$$= \frac{4ap}{2a^2 + 2p^2} = \frac{2ap}{a^2 + p^2}.$$

4. $$\frac{6x^2-4y^2+3z^2}{24}+\frac{6y^2-4z^2+3x^2}{24}+\frac{6z^2+4x^2+3y^2}{24}=\frac{13x^2+5y^2+5z^2}{24}$$

$$\frac{13x^2+5y^2+5z^2}{24}-\frac{24z^2-24x^2+12y^2}{24}=\frac{37x^2-7y^2-19z^2}{24}.$$

5. $$\frac{16+\frac{1}{4}-1+4}{16-\frac{1}{4}-1+1}=\frac{19\frac{1}{4}}{15\frac{3}{4}}=\frac{77}{63}=\frac{11}{9}.$$

6. $$\frac{15x^2+18ax-14a^2}{6}\times\frac{4x^2-2ax-a^2}{2}$$

$$=\frac{60x^4+42ax^3-107a^2x^2+10a^3x+14a^4}{12}.$$

7. $$\frac{(a-b)(a^2+ab+b^2)}{(a-b)^2}=\frac{a^2+ab+b^2}{a-b}=a+2b+\frac{3b^2}{a-b},$$ by division.

8. $$\frac{x^2-2xy+y^2+2xy}{x(x-y)}+\frac{y^3-xy^2}{x(x^2-y^2)}=\frac{x^2+y^2}{x(x-y)}+\frac{y^3-xy^2}{x(x^2-y^2)}$$

$$=\frac{x^3+xy^2+x^2y+y^3+y^3-xy^2}{x(x^2-y^2)}=\frac{x^3+x^2y+2y^3}{x(x^2-y^2)}.$$

9. $$\begin{array}{l} 5x^2+9x-2)\;60x^3-17x^2-4x+1\;(12x-25 \\ \qquad\qquad\quad \underline{60x^3+108x^2-24x} \\ \qquad\qquad\qquad\; -125x^2+20x+1 \\ \qquad\qquad\qquad\; \underline{-125x^2-225x+50} \\ \qquad\qquad\qquad\qquad\qquad 245x-49 \end{array}$$

and $$\frac{245x-49}{5x^2+9x-2}=\frac{49(5x-1)}{(x+2)(5x-1)}=\frac{49}{x+2}.$$

10. $$\begin{array}{l} x^4-9x^3+7x^2+9x-8)\;x^4+7x^3-9x^2-7x+8\;(1 \\ \qquad\qquad\qquad\qquad\qquad \underline{x^4-9x^3+7x^2+9x-8} \\ \qquad\qquad\qquad\qquad\qquad\quad 16x^3-16x^2-16x+16 \end{array}$$

[illegible]

$$\begin{array}{r} x^3-x^2-x-1 \\ \hline -3x^3-3x^2-3x-3 \\ -3x^3-3x^2-3x-3 \\ \hline \end{array}$$

Hence x^3-x^2-x-1 is the H. C. F.

11. $\frac{x^3}{x^3-1}+\frac{1}{\frac{1}{x}-\frac{1}{x}\cdot\frac{1}{1-x^3-1}}=\frac{x^3}{x^3-1}+\frac{-x^2}{-x^2+x^3-1+x}$

$=\frac{x^3}{x^3-1}+\frac{-x^2(x+1)}{x^3-1}=\frac{-x^2}{x^2-1}=\frac{x^2}{1-x^2}.$

12. $a+ab+b^2\left(a+ab+\frac{ab^2}{1-b}\right)=a+ab+b^2\left(\frac{a+ab-ab-ab^2+ab^2}{1-b}\right)$

$=a+ab+\frac{ab^2}{1-b}=\frac{a+ab-ab-ab^2+ab^2}{1-b}=\frac{a}{1-b}.$

13. $\left(l+\frac{1}{l}\right)\left(l-\frac{1}{l}\right)\left(l^2+\frac{1}{l^2}\right)=\left(l^2-\frac{1}{l^2}\right)\left(l^2+\frac{1}{l^2}\right)=l^4-\frac{1}{l^4}.$

14. $\frac{a+b+2}{ab+a+b+1}+\frac{1}{c+1}=\frac{ac+bc+2c+a+b+2+ab+a+b+1}{abc+ab+ac+a+bc+b+c+1}$

$=\frac{ab+ac+bc+2a+2b+2c+3}{abc+ab+ac+a+bc+b+c+1}$

If this fraction $=1$, then $ab+ac+bc+2a+2b+2c+3$

$=abc+ab+ac+a+bc+b+c+1$;

or, $ab+ac+bc+2a+2b+2c+3-ab-ac-a-bc-b-c-1=abc$,

or, $a+b+c+2=abc$.

15. $\dfrac{ax^3 - a^2x^2 - abx^2 - b^2x^2 + a^2bx + a^2b^2}{a^2x^2} \div x - a$

$$
\begin{array}{l}
x - a)\ ax^3 - a^2x^2 - abx^2 - b^2x^2 + a^2bx + a^2b^2\ (ax^2 - abx - b^2x - ab^2 \\
\quad \underline{ax^3 - a^2x^2} \\
\qquad\qquad - abx^2 - b^2x^2 + a^2bx \\
\qquad\qquad \underline{- abx^2 + a^2bx} \\
\qquad\qquad\qquad - b^2x^2 + a^2b^2 \\
\qquad\qquad\qquad \underline{- b^2x^2 + ab^2x} \\
\qquad\qquad\qquad\qquad - ab^2x + a^2b^2 \\
\qquad\qquad\qquad\qquad \underline{- ab^2x + a^2b^2}
\end{array}
$$

Hence the result is $\dfrac{ax^2 - abx - b^2x - ab^2}{a^2x^2}$, or, $\dfrac{1}{a} - \dfrac{b}{ax} - \dfrac{b^2}{a^2x} - \dfrac{b^2}{ax^2}$.

16. $\left(\dfrac{a}{bc} + \dfrac{b}{ac} + \dfrac{c}{ab}\right) \div \left(\dfrac{b}{ac} + \dfrac{c}{ab} + \dfrac{a}{bc}\right) = 1$

Also, since $s = \dfrac{a+b+c}{2}$

$s - a = \dfrac{b+c-a}{2}$; $s - b = \dfrac{a+c-b}{2}$; $s - c = \dfrac{a+b-c}{2}$.

Hence $\dfrac{s.(s-a) + (s-b)(s-c)}{bc} = \dfrac{\dfrac{(b+c)^2 - a^2}{4} + \dfrac{a^2 - (b-c)^2}{4}}{bc}$

$= \dfrac{b^2 + 2bc + c^2 - a^2 + a^2 - b^2 + 2bc - c^2}{4bc} = \dfrac{4bc}{4bc} = 1.$

17. $\dfrac{1}{1 + \dfrac{x}{a}} + \dfrac{1}{1 - \dfrac{x}{a}} + \dfrac{2}{1 + \dfrac{x^2}{a^2}} = \dfrac{a}{a+x} + \dfrac{a}{a-x} + \dfrac{2a^2}{a^2+x^2}$

$= \dfrac{a^2 - ax + a^2 + ax}{a^2 - x^2} + \dfrac{2a^2}{a^2 + x^2} = \dfrac{2a^2(a^2 + x^2) + 2a^2(a^2 - x^2)}{a^4 - x^4} = \dfrac{4a^4}{a^4 - x^4}$

18. $$\frac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}-\frac{2(a^2-b^2)}{a^2+b^2}=\frac{2(a^2+b^2)}{a^2-b^2}-\frac{2(a^2-b^2)}{a^2+b^2}$$

$$=\frac{2(a^4+2a^2b^2+b^4)-2(a^4-2a^2b^2+b^4)}{a^4-b^4}=\frac{8a^2b^2}{a^4-b^4}.$$

19. $$\frac{2ab-a^2-2ab-b^2}{2a(a+b)}+\frac{a^2+b^2}{2a(a-b)}=\frac{(a-b)(-a^2-b^2)+(a+b)(a^2+b^2)}{2a(a^2-b^2)}$$

$$=\frac{-a^3-ab^2+a^2b+b^3+a^3+ab^2+a^2b+b^3}{2a(a^2-b^2)}=\frac{2a^2b+2b^3}{2a(a^2-b^2)}=\frac{b(a^2+b^2)}{a(a^2-b^2)}.$$

20. $$\frac{a^2-ab+b^2}{(a-b)^3}\times\frac{(a+b)(a-b)}{a^2+b^2}=\frac{(a^2-ab+b^2)(a+b)}{(a-b)^2(a^2+b^2)}=\frac{a^3+b^3}{(a-b)^2(a^2+b^2)}.$$

21. $$\frac{2}{(x^2-1)^2}-\frac{1}{2(x-1)^2}+\frac{1}{x^2-1}=\frac{2}{(x^2-1)^2}-\frac{(x+1)^2}{2(x^2-1)^2}+\frac{x^2-1}{(x^2-1)^2}$$

$$=\frac{4-x^2-2x-1+2x^2-2}{2(x^2-1)^2}=\frac{x^2-2x+1}{2(x^2-1)^2}=\frac{1}{2(x+1)^2}.$$

22. $$\frac{(a+b)^2-c^2}{c^2-(a-b)^2}\div\frac{a+b+c}{b+c-a}=\frac{(a+b+c)(a+b-c)(b+c-a)}{(c+a-b)(c-a+b)(a+b+c)}=\frac{a+b-c}{a-b+c}.$$

23. $$\left(\frac{x^2}{x+1}+1-\frac{1}{x+1}\right)\div\left(\frac{x^2}{x-1}-x-\frac{1}{x-1}\right)$$

$$=\frac{x^2+x+1-1}{x+1}\div\frac{x^2-x^2+x-1}{x-1}=\frac{x^2+x}{x+1}\div 1=\frac{x(x+1)}{x+1}=x.$$

24. $$\left(\frac{\frac{a+b}{2}-a}{\frac{a+b}{2}-b}\right)^3-\frac{\frac{a+b}{2}-2a+b}{\frac{a+b}{2}+a-2b}=\left(\frac{b-a}{a-b}\right)^3-\frac{3b-3a}{3a-3b}=(-1)^3-(-1)$$

$$=-1+1=0.$$

25. $$\frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}+\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}$$

$$=\frac{a+b-c}{a+c+b}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{b+c+a}=\frac{b+c+a}{a+b+c}=1.$$

26. $\dfrac{x(x-4)(x^2-4)(x^2-4)}{x^2(x-2)(x-2)}=\dfrac{(x-4)(x+2)(x+2)}{x}=\dfrac{(x-4)(x+2)^2}{x}.$

27. $\dfrac{(a+1)(a-1)(a^3+1)(a^3-1)}{(a+1)(a+1)a^2(a-1)(a-1)}=\dfrac{(a^2-a+1)(a^2+a+1)}{a^2}=\dfrac{a^4+a^2+1}{a^2}.$

28. $\dfrac{1+x-x^2}{x^3}-\dfrac{1}{(x^2+1)^2}+\dfrac{(x^2+1)(x-1))}{(x^2+1)^2}-\dfrac{3}{x^2(x^2+1)^2}$

$$=\frac{1+x-x^2}{x^3}-\frac{x^3}{x^3(x^2+1)^2}+\frac{x^3(x^3-x^2+x-1)}{x^3(x^2+1)^2}-\frac{3x}{x^3(x^2+1)^2}$$

$$=\frac{(x^4+2x^2+1)(1+x-x^2)-x^3+x^3(x^3-x^2+x-1)-3x}{x^3(x^2+1)^2}$$

$$=\frac{1+x+x^2+2x^3-x^4+x^5-x^6-x^3+x^6-x^5+x^4-x^3-3x}{x^3(x^2+1)^2}$$

$$=\frac{1-2x+x^2}{x^3(x^2+1)^2}=\frac{(x-1)^2}{x^3(x^2+1)^2}.$$

29. $\dfrac{x^6-a^2x^4+a^4x^2-a^6}{a^3x^3}\div\dfrac{x^2-a^2}{ax}=\dfrac{x^4+a^4}{a^2x^2}=\dfrac{x^2}{a^2}+\dfrac{a^2}{x^2}.$

30. $\left\{\dfrac{a^2+2ab+b^2-a^2+2ab-b^2}{2(a^2-b^2)}+\dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{2b}$

$$=\left\{\frac{2ab+2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}=\frac{2b(a+b)(a-b)}{(a+b)(a-b)2b}=1.$$

31. $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc+b^2-2bc+c^2+c^2-2ac+a^2-2ab+b^2}{a^2+b^2+c^2}$

$$=\frac{3a^2+3b^2+3c^2}{a^2+b^2+c^2}=3.$$

32. $\dfrac{1+3x^2+2x^3}{(3-2x-7x^2)^4}-\dfrac{(1-x-3x^2)(3-2x-7x^2)}{(3-2x-7x^2)^4}$

$$=\frac{1+3x^2+2x^3-(3-5x-14x^2+13x^3+21x^4)}{(3-2x-7x^2)^4}$$

$$=\frac{-2+5x+17x^2-11x^3-21x^4}{(3-2x-7x^2)^4}.$$

33. $$\frac{x^4+2x^2y^2+y^4-x^4+2x^2y^2-y^4}{(x^2-y^2)(x^2+y^2)} \div \frac{x^2+2xy+y^2-x^2+2xy-y^2}{(x-y)(x+y)}$$
$$=\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)} \times \frac{(x-y)(x+y)}{4xy}=\frac{xy}{x^2+y^2}.$$

34. $$\frac{x^2-y^2}{y^2} \times \frac{y}{x-y}+\frac{x^3-y^3}{y^3} \times \frac{-y^2}{x^2+xy+y^2}=\frac{x+y}{y}+\frac{(x-y)(-1)}{y}=\frac{2y}{y}=2$$

35. $$\frac{a(a-b)}{(a-b)(a^2+ab+b^2)} \times \frac{a^2+ab+b^2}{a+b}+\frac{a^3-b^3}{a^3+b^3} \times \frac{a^2-ab+b^2}{a^2+ab+b^2}$$
$$=\frac{a}{a+b}+\frac{a-b}{a+b}=\frac{2a-b}{a+b}.$$

36. $$\frac{2-(x-1)}{4(x-1)^2}+\frac{(x-1)^2-4}{4(x-1)^2(x+1)}=\frac{3-x}{4(x-1)^2}+\frac{x^2-2x-3}{4(x-1)^2(x+1)}$$
$$=\frac{3x+3-x^2-x+x^2-2x-3}{4(x-1)^2(x+1)}=0.$$

37. $$\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$
$$=\frac{(a-b)(x-a)+bx}{abx(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$
$$=\frac{(ax+ab-a^2)(x-b)-ax(x-a)}{abx(a-b)(x-a)(x-b)}$$
$$=\frac{ax^2-abx+abx-ab^2-a^2x+a^2b-ax^2+a^2x}{abx(a-b)(x-c)(x-b)}=\frac{1}{x(x-a)(x-b)}.$$

38. $$\frac{s}{a}-\frac{a}{a}+\frac{s}{b}-\frac{b}{b}+\frac{s}{c}-\frac{c}{c}+\ldots \text{ to } n \text{ terms}$$
$$=\frac{s}{a}+\frac{s}{b}+\frac{s}{c}+\ldots \text{ to } n \text{ terms} - (1+1+1\ldots \text{ to } n \text{ terms})$$
$$=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right)-n.$$

39. $$\frac{x^4+x^2y^2-x^2y^2+y^4}{x^4-y^4} \times \frac{(x^2-y^2)^2}{x^4-2x^2y^2+y^4+x^4+2x^2y^2+y^4}$$
$$=\frac{(x^4+y^4)(x^2-y^2)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)\,2\,(x^4+y^4)}=\frac{x^2-y^2}{2(x^2+y^2)}.$$

40. $\dfrac{a+x+a-x}{a+x-a+x} \div \dfrac{a^2+x^2+a^2-x^2}{a^2+x^2-a^2+x^2} = \dfrac{2a}{2x} \times \dfrac{2x^2}{2a^2} = \dfrac{x}{a}.$

41. Since $x^3+\dfrac{1}{x^3}=\left(x+\dfrac{1}{x}\right)\left(x^2-1+\dfrac{1}{x^2}\right)$; and $\dfrac{1}{x^2}-x^2=\left(\dfrac{1}{x}+x\right)\left(\dfrac{1}{x}-x\right)$

the quotient is $x^2-1+\dfrac{1}{x^2}-3\left(\dfrac{1}{x}-x\right)+4$;

or, $x^2+3x+3-\dfrac{3}{x}+\dfrac{1}{x^2}$.

The quotient may also be obtained by long division as in p. 133 of the *Algebra*.

42. $\left(\dfrac{s}{s}-\dfrac{a}{s}\right)+\left(\dfrac{s}{s}-\dfrac{b}{s}\right)+\left(\dfrac{s}{s}-\dfrac{c}{s}\right)+\dots$ to n terms

$=(1+1+1\dots$ to n terms$)-\dfrac{1}{s}(a+b+c+\dots$to n terms$)$

$=n-\dfrac{s}{s}=n-1.$

43. $\dfrac{x^2+xy-xy+y^2}{x^2-y^2} \div \dfrac{x^4-x^2y^2+x^2y^2+y^4}{(x^2+y^2)(x^2-y^2)} = \dfrac{x^2+y^2}{x^2-y^2} \times \dfrac{(x^2+y^2)(x^2-y^2)}{x^4+y^4}$

$=\dfrac{(x^2+y^2)^2}{x^4+y^4}.$

44. $\dfrac{\dfrac{x^2+2xy+y^2-2xy}{(x+y)^2}}{\dfrac{x^2-2xy+y^2+2xy}{(x-y)^2}} \div \dfrac{(x-y)^2}{(x+y)^2} = \dfrac{(x^2+y^2)(x-y)^2}{(x+y)^2(x^2+y^2)} \times \dfrac{(x+y)^2}{(x-y)^2} = 1.$

45. $(a+b)(cd-1)=(1-ab)(c+d)$, or $acd-a+bcd-b=c-abc+d-abd$,

or $a+b+c+d=bcd+acd+abd+abc$

$$=abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

$\therefore \dfrac{a+b+c+d}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}}=abcd.$

46. $\dfrac{(p+q)^4}{(p-q)^4} \div \dfrac{(p+q)^3}{(p-q)^3} = \dfrac{p+q}{p-q}.$

47. $$\frac{1-2x}{3(x^2-x+1)}+\frac{1}{6(x+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{(1-2x)(2x+2)+(x^2-x+1)}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{2x+2-4x^2-4x+x^2-x+1}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{3-3x-3x^2}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}=\frac{1-x-x^2}{2(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{x^2-x^3-x^4+1-x-x^2+x^4+x^3+x+1}{2(x^3+1)(x^2+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

48. $$\frac{1}{x+\dfrac{z}{yz+1}}\div\frac{y}{xy+1}-\frac{1}{y(xyz+x+z)}$$

$$=\frac{yz+1}{xyz+x+z}\times\frac{xy+1}{y}-\frac{1}{y(xyz+x+z)}=\frac{xy^2z+xy+yz+1-1}{y(xyz+x+z)}=1.$$

49. $$\frac{1}{a-x}+\frac{x}{(a-x)^2}-\left(\frac{1}{a-y}+\frac{y}{(a-y)^2}\right)=\frac{a}{(a-x)^2}-\frac{a}{(a-y)^2}$$

And $$\frac{1}{(a-y)(a-x)^2}-\frac{1}{(a-y)^2(a-x)}=\frac{(a-y)-(a-x)}{(a-x)^2(a-y)^2}$$

Hence original fraction $$=\frac{a(a-y)^2-a(a-x)^2}{(a-y)-(a-x)}=a(a-y)+a(a-x)$$

$$=2a^2-ax-ay.$$

50. $$\frac{3}{abc}\div\frac{a+b-c}{abc}-\frac{3-a-b-c}{a+b-c}=\frac{3}{a+b-c}-\frac{3-a-b-c}{a+b-c}=\frac{a+b+c}{a+b-c}.$$

51. $$\frac{a+\dfrac{b^2}{b+a}}{a-\dfrac{b^2}{b-a}}(a^6-b^6)=\frac{(ab+a^2+b^2)(b-a)}{(ab-a^2-b^2)(b+a)}\cdot(a^6-b^6)$$

$$=\frac{b^3-a^3}{-(a^3+b^3)}\cdot(a^6-b^6)=\frac{a^3-b^3}{a^3+b^3}\cdot(a^3+b^3)(a^3-b^3)=(a^3-b^3)^2.$$

LXX.

1. $\left.\begin{aligned}6x+21y&=123\\6x+\ \ 8y&=\ \ 84\end{aligned}\right\}$ subtracting, $13y=39$, $y=3$, etc.

2. $\left.\begin{aligned}45x+72y&=909\\45x+10y&=475\end{aligned}\right\}$ subtracting, $62y=434$, $y=7$, etc.

3. $\left.\begin{aligned}26x+34y&=378\\26x+13y&=273\end{aligned}\right\}$ subtracting, $21y=105$, $y=5$, etc.

4. $\left.\begin{aligned}14x+9y&=156\\14x+4y&=116\end{aligned}\right\}$ subtracting, $5y=40$, $y=8$, etc.

5. $\left.\begin{aligned}3x+45y&=147\\3x+\ \ 7y&=\ \ 71\end{aligned}\right\}$ subtracting, $38y=76$, $y=2$, etc.

6. $\left.\begin{aligned}105x+133y&=924\\105x+\ \ 51y&=678\end{aligned}\right\}$ subtracting, $82y=246$, $y=3$, etc.

7. $\left.\begin{aligned}6x+\ \ 4y&=\ \ 236\\6x+30y&=1146\end{aligned}\right\}$ subtracting, $-26y=-910$, $y=35$, etc.

8. $\left.\begin{aligned}156x+108y&=420\\156x+\ \ 87y&=399\end{aligned}\right\}$ subtracting, $21y=21$, $y=1$, etc.

9. $\left.\begin{aligned}504x+98y&=2310\\504x+56y&=2184\end{aligned}\right\}$ subtracting, $42y=126$, $y=3$, etc.

LXXI.

1. $\left.\begin{aligned}6x+21y&=156\\6x-10y&=\ \ 32\end{aligned}\right\}$ subtracting, $31y=124$, $y=4$, etc.

2. $\left.\begin{aligned}105x-60y&=825\\105x-91y&=763\end{aligned}\right\}$ subtracting, $31y=62$, $y=2$, etc.

3. $\left.\begin{aligned}x+y&=96\\x-y&=\ \ 2\end{aligned}\right\}$ subtracting, $2y=94$, $y=47$, etc.

4. $\left.\begin{aligned} 28x+63y&=553 \\ 28x-68y&=160 \end{aligned}\right\}$ subtracting, $131y=393$, $y=3$, etc.

5. $\left.\begin{aligned} 7x+133y&=679 \\ 7x-\ 53y&=121 \end{aligned}\right\}$ subtracting, $186y=558$, $y=3$, etc.

6. $\left.\begin{aligned} 87x-42y&=525 \\ 87x-56y&=497 \end{aligned}\right\}$ subtracting, $14y=28$, $y=2$, etc.

7. $\left.\begin{aligned} 342x-426y&=1284 \\ 342x-978y&=\ 732 \end{aligned}\right\}$ subtracting, $552y=552$, $y=1$, etc.

8. $\left.\begin{aligned} 516x+\ 24y&=3192 \\ 516x-731y&=\ 172 \end{aligned}\right\}$ subtracting, $755y=3020$, $y=4$, etc.

9. $\left.\begin{aligned} 65x+117y&=2444 \\ 65x-\ 10y&=\ 285 \end{aligned}\right\}$ subtracting, $127y=2159$, $y=17$, etc.

LXXII.

1. $\left.\begin{aligned} 12x-21y&=\ 66 \\ 12x-28y&=-4 \end{aligned}\right\}$ subtracting, $7y=70$, $y=10$, etc.

2. $\left.\begin{aligned} 9x-\ 5y&=\ 52 \\ 9x-24y&=-24 \end{aligned}\right\}$ subtracting, $19y=76$, $y=4$, etc.

3. $\left.\begin{aligned} 51x+\ 9y&=\ 171 \\ 51x-272y&=-391 \end{aligned}\right\}$ subtracting, $281y=562$, $y=2$, etc.

4. $\left.\begin{aligned} 21x+49y&=\ 546 \\ 21x-57y&=-408 \end{aligned}\right\}$ subtracting, $106y=954$, $y=9$, etc.

5. $\left.\begin{aligned} 35x-21y&=\ 28 \\ 35x-60y&=-50 \end{aligned}\right\}$ subtracting, $39y=78$, $y=2$, etc.

6. $\left.\begin{aligned} 6x+4y&=\ 78 \\ 6x-9y&=-39 \end{aligned}\right\}$ subtracting, $13y=117$, $y=9$, etc.

7. $\left.\begin{array}{r} 26x - 65y = -273 \\ 26x - 8y = 240 \end{array}\right\}$ subtracting, $-57y = -513$, $y = 9$, etc.

8. $\left.\begin{array}{r} 105x - 135y = -195 \\ 105x - 49y = 63 \end{array}\right\}$ subtracting, $-86y = -258$, $y = 3$, etc.

9. $\left.\begin{array}{r} 21y + 36x = 528 \\ 21y - 133x = 21 \end{array}\right\}$ subtracting, $169x = 507$, $x = 3$, etc.

LXXIII.

1. $\left.\begin{array}{r} 6x + 9y = 24 \\ 6x + 14y = 14 \end{array}\right\}$ subtracting, $-5y = 10$, $y = -2$, etc.

2. $\left.\begin{array}{r} 15x - 6y = 153 \\ 38x - 6y = 360 \end{array}\right\}$ subtracting, $-23x = -207$, $x = 9$, etc.

3. $\left.\begin{array}{r} 6x - 10y = 102 \\ 6x + 21y = 9 \end{array}\right\}$ subtracting, $-31y = 93$, $y = -3$.

4. $\left.\begin{array}{r} 35y - 15x = 695 \\ 35y + 14x = 637 \end{array}\right\}$ subtracting, $-29x = 58$, $x = -2$, etc.

5. $\left.\begin{array}{r} 8x + 18y = 212 \\ 8x + 17y = 198 \end{array}\right\}$ subtracting, $y = 14$, etc.

6. $\left.\begin{array}{r} 18x - 63y = 72 \\ 18x - 8y = -38 \end{array}\right\}$ subtracting, $-55y = 110$, $y = -2$, etc.

7. $\left.\begin{array}{r} 17x + 12y = 59 \\ 57x - 12y = 459 \end{array}\right\}$ adding, $74x = 518$, $x = 7$, etc.

8. $\left.\begin{array}{r} 24x + 9y = 9 \\ 12x + 9y = 3 \end{array}\right\}$ subtracting, $12x = 6$, $x = \frac{1}{2}$, etc.

9. $\left.\begin{array}{r} 238x - 966y = -1442 \\ 238x - 221y = -697 \end{array}\right\}$ subtracting, $-745y = -745$, $y = 1$, etc.

LXXIV.

1. $\left.\begin{array}{l}3x+2y=42\\2x+3y=48\end{array}\right\}$, $\left.\begin{array}{l}6x+4y=\ \ 84\\6x+9y=144\end{array}\right\}$, $-5y=-60$, $y=12$, etc.

2. $\left.\begin{array}{l}30x+y=630\\20y-x=580\end{array}\right\}$, $\left.\begin{array}{l}600x+20y=12600\\\ \ 20y-\ x\ =\ \ \ \ 580\end{array}\right\}$, $601x=12020$, $x=20$, etc.

3. $\left.\begin{array}{l}x+49y=1757\\y+49x=2093\end{array}\right\}$, $\left.\begin{array}{l}49x+2401y=86093\\49x+\ \ \ \ y\ \ =\ \ 2093\end{array}\right\}$, $2400y=84000$, $y=35$, etc.

4. $\left.\begin{array}{l}x+y+15=30\\x-y+14=19\end{array}\right\}$, $\left.\begin{array}{l}x+y=15\\x-y=\ \ 5\end{array}\right\}$, $2x=20$, $x=10$, etc.

5. $\left.\begin{array}{l}14x+5y\ \ =\ \ \ \ 826\\39x-14y=-1609\end{array}\right\}$, $\left.\begin{array}{l}196x+70y=\ \ 11564\\195x-70y=-8045\end{array}\right\}$,

$391x=3519$, $x=9$, etc.

6. $\left.\begin{array}{l}6x+9y=150-5y\\8y-6x=\ \ 9x+12\end{array}\right\}$, $\left.\begin{array}{l}6x+14y=150\\8y-15x=\ \ 12\end{array}\right\}$, $\left.\begin{array}{l}30x+70y=750\\16y-30x=\ \ 24\end{array}\right\}$,

$86y=774$, etc.

7. $\left.\begin{array}{l}7x-y+2=35\\12y-x-10=9\end{array}\right\}$, $\left.\begin{array}{l}7x-y=33\\12y-x=19\end{array}\right\}$, $\left.\begin{array}{l}7x-\ \ y=\ \ 33\\84y-7x=133\end{array}\right\}$, $83y=166$, etc.

8. $\left.\begin{array}{l}x+32=2y-48\\12x+12y+20y=30x-15y+2100\end{array}\right\}$, $\left.\begin{array}{l}x-\ \ 2y=-80\\47y-18x=2100\end{array}\right\}$,

$\left.\begin{array}{l}18x-36y=-1440\\47y-18x=\ \ \ \ 2100\end{array}\right\}$, adding, $11y=660$, $y=60$, etc.

9. $\left.\begin{array}{l}15x-25y+30=4x+2y\\\ \ 96-\ \ 3x+6y=6x+4y\end{array}\right\}$, $\left.\begin{array}{l}11x-27y=-30\\\ \ 2y-\ \ 9x=-96\end{array}\right\}$,

$\left.\begin{array}{l}99x-243y=-270\\22y-\ \ 99x=-1056\end{array}\right\}$, adding, $-221y=-1326$, $y=6$, etc.

10. $\left.\begin{aligned} x+2+24y &= 93 \\ y+5+40x &= 768 \end{aligned}\right\}, \left.\begin{aligned} x+24y &= 91 \\ 40x+y &= 763 \end{aligned}\right\}, \left.\begin{aligned} 40x+960y &= 3640 \\ 40x+y \quad &= 763 \end{aligned}\right\},$

$959y = 2877$, etc.

11. $\left.\begin{aligned} 2x-y+21x &= 14y-42 \\ 6y+18+5y-5x &= 60x-240 \end{aligned}\right\}, \left.\begin{aligned} 23x-15y &= -\ 42 \\ 11y-65x &= -258 \end{aligned}\right\},$

$\left.\begin{aligned} 253x-165y &= -\ 462 \\ 165y-975x &= -3870 \end{aligned}\right\}$ adding, $-722x = -4332$, $x = 6$, etc.

12. $\left.\begin{aligned} 12x-24-200+20x &= 15y-150 \\ 16y+32 &= 12x+3y+39 \end{aligned}\right\}, \left.\begin{aligned} 32x-15y &= 74 \\ 13y-12x &= 7 \end{aligned}\right\},$

$\left.\begin{aligned} 96x-45y &= 222 \\ 104y-96x &= 56 \end{aligned}\right\}$, adding, $59y=278$, $y=\dfrac{278}{59}$, etc.

13. $\left.\begin{aligned} 5x-6y+39x &= 52y-26 \\ 10x+12y-9x+6y &= 24y-24 \end{aligned}\right\}, \left.\begin{aligned} 44x-58y &= -26 \\ x-\ 6y &= -24 \end{aligned}\right\}$

$\left.\begin{aligned} 22x-29y \quad &= -\ 13 \\ 22x-132y &= -528 \end{aligned}\right\}$, $103y = 515$, $y = 5$, etc.

14. $\left.\begin{aligned} 15x-9\ -\ 9x+57 &= 24\ -6y+2x \\ 16x+8y-18x+14 &= 12y+36-4x-5y \end{aligned}\right\}, \left.\begin{aligned} 4x+6y &= -24 \\ 2x+\ y &= 22 \end{aligned}\right\},$

$\left.\begin{aligned} 2x+3y &= -12 \\ 2x+\ y &= 22 \end{aligned}\right\}$, $2y= -34$, $y= -17$, etc.

15. $\left.\begin{aligned} 4x+5y &= 40x-40y \\ 4x-2y+12y &= 3 \end{aligned}\right\}, \left.\begin{aligned} 45y-36x &= 0 \\ 4x+10y &= 3 \end{aligned}\right\}, \left.\begin{aligned} 5y-\ 4x &= 0 \\ 4x+10y &= 3 \end{aligned}\right\}$, $15y=3$, etc.

LXXV.

1. $\left.\begin{aligned} mpx+npy &= ep \\ mpx+mqy &= fm \end{aligned}\right\}$ subtracting $(np-mq)y = ep-fm$, etc.

2. $\left.\begin{aligned} adx+bdy &= cd \\ adx-aey &= af \end{aligned}\right\}$ subtracting $(bd+ae)\,y = cd-af$, etc.

3. $\left.\begin{array}{l} acx - bcy = cm \\ acx + aey = an \end{array}\right\}$, $y(ae + bc) = an - cm$, etc.

4. $\left.\begin{array}{l} cx - dy = 0 \\ cx + cy = ce \end{array}\right\}$ $(c + d)\,y = ce$, etc.

5. $\left.\begin{array}{l} mm'x - m'ny = m'r \\ mm'x + mn'y = mr' \end{array}\right\}$, $(mn' + m'n)\,y = mr' - m'r$, etc.

6. $\left.\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$, $2x = e + b$, $x = \dfrac{a + b}{2}$, etc.

7. $\left.\begin{array}{l} adx + bdy = cd \\ adx + afy = ac^2 \end{array}\right\}$, $(af - bd)\,y = ac^2 - cd$, etc.

8. $\left.\begin{array}{l} abdx + cd^2y = 2d \\ abdx - bcdy = d - b \end{array}\right\}$, $cdy\,(d + b) = d + b$, etc.

9. $\left.\begin{array}{l} 3a^2 + ax = b^2 + by \\ ax + 2by = d \end{array}\right\}$, $\left.\begin{array}{l} ax - by = b^2 - 3a^2 \\ ax + 2by = d \end{array}\right\}$, $3by = 3a^2 - b^2 + d$, etc.

10. $\left.\begin{array}{l} bcx - cy = -2b \\ b^3cy + ac^3 - ab^3 = 2b^4 + bc^4x \end{array}\right\}$, $\left.\begin{array}{l} bc^4x - c^4y = -2bc^3 \\ b^3cy - bc^4x = 2b^4 - ac^3 + ab^3 \end{array}\right\}$,

$$b^3cy - c^4y = 2b^4 - ac^3 + ab^3 - 2bc^3,$$

$$y = \frac{2b^4 - ac^3 + ab^3 - 2bc^3}{c\,(b^3 - c^3)} = \frac{2b + a}{c}, \text{ etc.}$$

11. $\left.\begin{array}{l} bx + bc - b^2 + cx + c^2 - bc + ay + a^2 = 2a^2 \\ a^3y = (b + c)^2\,(b - c)\,x \end{array}\right\}$,

$$\left.\begin{array}{l} (b + c)\,x + ay = a^2 + b^2 - c^2 \\ (b + c)\,(b^2 - c^2)\,x - a^3y = 0 \end{array}\right\}, \quad \left.\begin{array}{l} a^2(b + c)x + a^3y = a^2(a^2 + b^2 - c^2) \\ (b + c)\,(b^2 - c^2)\,x - a^3y = 0 \end{array}\right\};$$

$$(b + c)\,(a^2 + b^2 - c^2)\,x = a^2\,(a^2 + b^2 - c^2)\ ;\ x = \frac{a^2}{b + c}, \text{ etc.}$$

12. $$\left.\begin{array}{l} 3x + 5y = \dfrac{8b^2m - 2bm^2}{b^2 - m^2} \\ (b^2 - m^2)x + (b + c + m)\,my = b^2m + 2bm^2 + \dfrac{bcm^2}{b + m} \end{array}\right\}$$

$$\left.\begin{array}{l} 3(b^2-m^2)x+5(b^2-m^2)y=8b^2m-2bm^2 \\ 3(b^2-m^2)x+3(bm+cm+m^2)=3b^2m+6bm^2+\dfrac{3bcm^2}{b+m} \end{array}\right\}$$

$$(5b^2-3bm-3cm-8m^2)y=5b^2m-8bm^2-\frac{3bcm^2}{b+m}$$

$$=\frac{5b^3m+5b^2m^2-8b^2m^2-8bm^3-3bcm^2}{b+m}$$

$$=\frac{bm(5b^2-3bm-3cm-8m^2)}{b+m}, \text{ etc.}$$

LXXVI.

1. $\left.\begin{array}{l} \dfrac{4}{x}+\dfrac{8}{y}=40 \\ \dfrac{4}{x}+\dfrac{3}{y}=20 \end{array}\right\}$ subtracting, $\dfrac{5}{y}=20$; $y=\dfrac{1}{4}$, etc.

2. $\left.\begin{array}{l} \dfrac{3}{x}+\dfrac{6}{y}=3a \\ \dfrac{3}{x}+\dfrac{4}{y}=b \end{array}\right\}$ subtracting, $\dfrac{2}{y}=3a-b$; $y=\dfrac{2}{3a-b}$, etc.

3. $\left.\begin{array}{l} \dfrac{ab}{x}+\dfrac{b^2}{y}=bc \\ \dfrac{ab}{x}+\dfrac{a^2}{y}=ad \end{array}\right\}$ subtracting, $\dfrac{b^2-a^2}{y}=bc-ad$; $y=\dfrac{b^2-a^2}{bc-ad}$, etc.

4. $\left.\begin{array}{l} \dfrac{a}{x}+\dfrac{b}{y}=m \\ \dfrac{a}{x}-\dfrac{b}{y}=n \end{array}\right\}$ subtracting, $\dfrac{2b}{y}=m-n$; $y=\dfrac{2b}{m-n}$, etc.

5. $\left.\begin{array}{l} \dfrac{56}{x}+\dfrac{40}{y}=152 \\ \dfrac{56}{x}-\dfrac{21}{y}=\ \ 49 \end{array}\right\}$ subtracting, $\dfrac{61}{y}=103$; $y=\dfrac{61}{103}$, etc.

6. Multiply the second equation by 4, and we get

$$\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{14}{3x}-\frac{2}{5y}&=12\end{aligned}\right\}\text{ adding, } \frac{19}{3x}=19\,;\ x=\frac{1}{3},\text{ etc.}$$

7. $$\left.\begin{aligned}\frac{4}{ax}+\frac{6}{by}&=10\\ \frac{15}{ax}-\frac{6}{by}&=9\end{aligned}\right\}\text{ adding, } \frac{19}{ax}=19\,;\ x=\frac{1}{a},\text{ etc.}$$

8. Multiply the first equation by $\frac{n^2}{m}$, and we get

$$\left.\begin{aligned}\frac{n}{x}+\frac{n^3}{m^2y}&=\frac{mn^2+n^3}{m}\\ \frac{n}{x}+\frac{m}{y}\ &=m^2+n^2\end{aligned}\right\}\text{ subtracting, } \frac{n^3-m^3}{m^2y}=\frac{n^3-m^3}{m}\,;\ y=\frac{1}{m},\text{ etc.}$$

LXXVII.

1. From the first and second equations we get

$$\left.\begin{aligned}5x+7y-2z&=13\\ 16x+6y+2z&=34\end{aligned}\right\}\text{ adding, } 21x+13y=47\ (1)$$

From the first and third equations we get

$$\left.\begin{aligned}25x+35y-10z&=65\\ x-\ 4y+10z&=23\end{aligned}\right\}\text{ adding, } 26x+31y=88\ (2)$$

Then from (1) and (2) we find $x=1$, $y=2$; and hence $z=3$.

2. From the first and second equations we get

$$\left.\begin{aligned}5x+3y-6z&=4\\ 9x-3y+6z&=24\end{aligned}\right\}\text{ adding, } 14x=28\,;\ x=2$$

From the second and third we get $2x+y=6$; $\therefore y=2$, etc.

3. From the first and second equations we get

$$\left.\begin{aligned}15x-9y+6z&=63\\ 16x-2y-6z&=6\end{aligned}\right\}\text{ adding, } 31x-11y=69\ (1)$$

From the first and third, subtracting, $3x - 6y = -18$ (2)

Then from (1) and (2) we find $x = 4$, $y = 5$; and $\therefore z = 8$.

4. From the first and second equations we get

$$\left.\begin{aligned} 4x - 5y + 2z &= 6 \\ 4x + 6y - 2z &= 40 \end{aligned}\right\} \text{ adding, } 8x + y = 46 \text{ (1)}$$

From the second and third equations we get

$$\left.\begin{aligned} 6x + 9y - 3z &= 60 \\ 7x - 4y + 3z &= 35 \end{aligned}\right\} \text{ adding, } 13x + 5y = 95 \text{ (2)}$$

Then from (1) and (2) we find $x = 5$, $y = 6$; and $\therefore z = 8$.

5. From the second and first equations we get

$$\left.\begin{aligned} 5x + 4y + 3z &= 22 \\ 3x + 3y + 3z &= 18 \end{aligned}\right\} \text{ subtracting, } 2x + y = 4 \text{ (1)}$$

From the third and second equations we get

$$\left.\begin{aligned} 15x + 10y + 6z &= 53 \\ 10x + 8y + 6z &= 44 \end{aligned}\right\} \text{ subtracting, } 5x + 2y = 9 \text{ (2)}$$

Then from (1) and (2) we find $x = 1$, $y = 2$; and $\therefore z = 3$.

6. From the first and second equations we get

$$\left.\begin{aligned} 8x + 4y - 3z &= 6 \\ 3x + 9y - 3z &= 21 \end{aligned}\right\} \text{ subtracting, } 5x - 5y = -15 \text{ (1)}$$

From the second and third equations we get

$$\left.\begin{aligned} 4x + 12y - 4z &= 28 \\ 4x - 5y + 4z &= 8 \end{aligned}\right\} \text{ adding, } 8x + 7y = 36 \text{ (2)}$$

Then from (1) and (2) we find $x = 1$, $y = 4$; and $\therefore z = 6$.

7. From the second and first equations we get

$$\left.\begin{aligned} 8x + 4y + 2z &= 50 \\ 2x + 2y + 2z &= 60 \end{aligned}\right\} \text{ subtracting, } 6x + 2y = -10 \text{ (1)}$$

From the third and first equations we get

$$\left.\begin{aligned} 27x+9y+3z&=64 \\ 3x+3y+3z&=90 \end{aligned}\right\} \text{ subtracting, } 24x+6y=-26 \ (2)$$

Then from (1) and (2) we find $x=\frac{2}{3}$, $y=-7$; and $\therefore z=36\frac{1}{3}$

8. From the first and second equations we get

$$\left.\begin{aligned} 4x-3y+\quad z&=9 \\ 27x+3y-15z&=48 \end{aligned}\right\} \text{ adding, } 31x-14z=57 \ (1)$$

From the second and third equations we get

$$\left.\begin{aligned} 36x+4y-20z&=64 \\ x-4y+\ 3z&=2 \end{aligned}\right\} \text{ adding, } 37x-17z=66 \ (2)$$

Then from (1) and (2) we find $x=5$, $z=7$; and $\therefore y=6$.

9. From the first and second equations we get

$$\left.\begin{aligned} 24x+10y-\ 8z&=58 \\ 65x-10y+25z&=290 \end{aligned}\right\} \text{ adding, } 89x+17z=348 \ (1)$$

From the first and third equations we get

$$\left.\begin{aligned} 12x+5y-4z&=29 \\ 85x-5y-5z&=75 \end{aligned}\right\} \text{ adding, } 97x-9z=104 \ (2)$$

Then from (1) and (2) we find $x=2$, $z=10$; and $\therefore y=9$.

10. Subtracting the second equation from the first, we get $2y=20$; $y=10$.

Adding the third equation to the second, we get $2z=10$; $z=5$.

LXXVIII.

1. Let x and y be the numbers; then $x+y=28$; $x-y=4$, etc.
2. Let x and y be the numbers; then $x+y=256$; $x-y=10$, etc.
3. Let x and y be the numbers; then $x+y=13{\cdot}5$; $x-y=1$, etc.

4. Let x and y be the numbers; then $7x+5y=332$;
$51x-51y=408$, etc.

5. Let x be the age of the father, and y the age of the son.
Then $x-7=4(y-7)$ and $x+7=2(y+7)$, etc.

6. Let x, y, z be the numbers; then $x+y=70, x+z=80, y+z=90$, etc.

7. Let x and y be the sums contributed by A and B;
then $400-(x+y)$ is the sum contributed by C.
Then $y=2x+20$, and $400-(x+y)=x+y$, etc.

8. Let x be A's money, and y B's money in shillings.
Then $3(x-10)=y+10$, and $x+10=2(y-10)$, etc.

9. Let x and y be A's and B's shares; then $760-(x+y)$ is C's share.
Then $x+y=760-x-y+240$; $y+760-x-y=x+360$, etc.

10. Let x and y be the numbers; then $\frac{x+y}{2}=24$; $\frac{x-y}{2}=17$, etc.

11. Let x be the greater number; and y the less; then
$\frac{x}{y}=4+\frac{3}{y}$; $\frac{x+y+38}{x}=2+\frac{2}{x}$, etc.

12. Let x be the first part, y the second, and $\therefore$ $144-(x+y)$ the third.
Then $\frac{x}{y}=3+\frac{2}{y}$; $\frac{144-x-y}{x+y}=2+\frac{6}{x+y}$, etc.

13. Let x be A's money and y B's money in pounds.
Then $x+\frac{y}{2}=120$; $\frac{2x}{3}+y=120$, etc.

14. Let x be the age of the father, y the age of the son.
Then $x-12=y+12$; $x+12=3(y-12)$, etc.

15. Let x be the greater number, and y the less.
Then $3x = 2x + 10$, and $2x + 3y = 24$, etc.

16. Let x be the age of the father, and y the age of the son.
Then $x + y = \frac{1}{2}(x + y + 50)$; $x - y = \frac{1}{3}(x + y + 40)$, etc.

17. Let x be the greater number, and y the less.
Then $\frac{y}{x} = \cdot 21 + \frac{\cdot 0157}{x}$; $\frac{x}{y} = 4 + \frac{\cdot 742}{y}$, etc.

18. Let x be the cost of a barrel of beer, y of a barrel of porter in pounds.
Then $6x + 10y = 51$; $3x + 7y = 32\frac{1}{10}$. Hence we find $x = 3$, and hence 10 barrels of beer can be bought for £30.

19. Let x be the cost of 1 lb. of tea, y of 1 lb. of coffee in pence.
Then $7x + 5y = 352$, and $4x + 9y = 324$, etc.

20. Let x be the cost of a horse, y of a cow in pounds.
Then $12x + 14y = 380$, and $5x + 3y = 130$, etc.

21. Let x be the cost of a yard of silk, y of a yard of cloth in pence.
Then $8x + 19y = 4370$, and $20x + 16y = 6200$, etc.

22. In one day 10 men and 6 women earn £3, 3s.
In one day 4 men and 8 women earn £2, 2s.
Let x be the daily earnings of a man, y of a woman in shillings.
Then $10x + 6y = 63$, and $4x + 8y = 42$, etc.

23. Let x be the number at £37; and y the number at £45.
Then $37x + 45y = 4220$; and $x + y = 100$, etc.

24. Let $10x + y$ be the number.
Then $x + y = 8$; and $10x + y + 36 = 10y + x$, etc.

25. Let $10x + y$ be the number.
Then $x + y = 10$; and $10x + y + 54 = 10y + x$, etc.

26. Let $10x + y$ be the number.
Then $x + y = 9$; and $10x + y + 9 = 10y + x$, etc.

27. Let $10x+y$ be the number.

Then $x+y=6$; and $\frac{10x+y}{x+y}=4$, etc.

28. Let $10x+y$ be the number.

Then $x+y=9$; and $\frac{10x+y}{x+y}=5$, etc.

29. Let $10x+y$ be the number.

Then $\frac{10x+y}{x+y}=7$; and $\frac{10y+x-12}{x-y}=9$, etc.

30. Let $10x+y$ be the number.

Then $\frac{10x+y}{x+y}=6+\frac{3}{x+y}$; and $\frac{10y+x}{x+y}=4+\frac{9}{x+y}$

or, $10x+y=6x+6y+3$; and $10y+x=4x+4y+9$, etc.

31. Let $10x+y$ be the number.

Then $\frac{10x+y}{x+y-2}=5+\frac{1}{x+y-2}$; $\frac{10y+x}{x+y+2}=5+\frac{8}{x+y+2}$, etc.

32. Let $10x+y$ and $10y+x$ be the numbers.

Then $10x+y+9=10y+x$, and $10x+y+10y+x=33$, etc.

33. Let $100x+10x+x$ be the number.

Then $100x+10x+x=37x^2$; $\therefore x=3$, etc.

34. Let $100x+10y+z$ be the number.

Then $y=2z$; $x+z=9$; $x+y+z=17$.

Hence $x=9-z$; and $\therefore 9-z+2z+z=17$; $z=4$, etc.

35. Let $100x+10y+z$ be the number.

Then $x+y+z=21$; $x+y=z+3$;

$100x+10y+z+198=100z+10y+x$, etc.

36. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+7}{y}=2$, and $\frac{x}{y-1}=1$, etc.

37. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$, etc.

38. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+1}=\frac{1}{3}$, etc.

39. Let $\frac{x}{y}$ be the fraction. Then $x+1=y$, and $x=\frac{1}{2}(y+1)$, etc.

40. Let $\frac{x}{y}$ be the fraction. Then $\frac{x-3}{y-3}=\frac{1}{4}$, and $\frac{x+5}{y+5}=\frac{1}{2}$, etc.

41. Let $\frac{x}{y}$ be the fraction. Then $\frac{x}{y+4}=\frac{7}{9}$, and $\frac{x-15}{y}=\frac{20}{41}$, etc.

42. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+17}=\frac{1}{3}$, etc.

43. Let x and y be the sums invested.

Then $\frac{x \times 5}{100}=$ income on first investment.

and $\frac{y \times 4}{100}=$ income on second investment.

$\therefore \frac{x \times 5}{100}=\frac{y \times 4}{100}+10$; and $x+y=2000$, etc.

44. Let x be the sum invested, and y the rate of interest.

$$\text{Then } x+\frac{x \times y \times 10}{100 \times 12}=5250$$

$$\text{and } x+\frac{x \times y \times 18}{100 \times 12}=5450.$$

$$\text{Hence } \left.\begin{aligned} 120x+xy&=630000 \\ \text{and } 200x+3xy&=1090000 \end{aligned}\right\}, \quad \left.\begin{aligned} 360x+3xy&=1890000 \\ 200x+3xy&=1090000 \end{aligned}\right\},$$

$160x=800000$, etc.

45. Let x be the sum invested, and y the rate of interest.

$$\text{Then } x+\frac{x \times y \times 6}{100}=5200.$$

$$\text{and } x+\frac{x \times y \times 10}{100}=6000$$

$$\text{Hence } \left.\begin{aligned} 50x+3xy&=260000 \\ 10x+xy&=60000 \end{aligned}\right\}, \quad \left.\begin{aligned} 50x+3xy&=260000 \\ 30x+3xy&=180000 \end{aligned}\right\},$$

$20x=80000$, etc.

46. Let x be the number of quarts of the first, y of the second.

Then $\left.\begin{aligned} x+y&=50 \\ 36x+20y&=30\times 50 \end{aligned}\right\}$, $\left.\begin{aligned} 36x+36y&=1800 \\ 36x+20y&=1500 \end{aligned}\right\}$, $16y=300$, etc.

47. Let x be the number of lbs. of the cheaper, y of the dearer.

Then $x+y=30$

and $20(x+y)=14x+18y+10x$, etc.

48. Let x be the rate of the rowing in miles an hour in still water, y the rate of the stream in miles an hour.

Then $x+y=12$, $x-y=6$, etc.

49. Let x be the distance in miles, y the rate in miles an hour of pulling.

Then $x=1\frac{2}{3}(y+4)$

and $x-3=4\frac{1}{2}(y-4)$, etc.

50. Let x be the number of leaps the hare takes, and y the length of each in feet.

Then $\frac{5x}{6}$ is the number of leaps the dog takes, and $\frac{9y}{7}$ the length of each in feet.

Then $50y+xy=\frac{5x}{6}\times\frac{9y}{7}$

Divide by y; then $50+x=\frac{15x}{14}$; $x=700$.

51. Let x be the number of leaps the dog takes, and y the length of each in feet.

Then $\frac{4x}{3}$ is the number of leaps the hare takes, and $\frac{2y}{3}$ is the length of each in feet.

Then $50y+\frac{4x}{3}\times\frac{2y}{3}=xy$.

Divide by y; then $50+\frac{8x}{9}=x$; $x=450$, etc.

52. Let x be the number of apples, and y the number of pears.

Then $\frac{x}{4}+\frac{y}{5}=30$; and $\frac{x}{8}+\frac{y}{15}=13$, etc.

53. Let x be the number of men, y the reckoning of each in shillings.

$$\left.\begin{array}{l}\text{Then } (x+3)(y-1)=xy \\ \text{and } (x-2)(y+1)=xy\end{array}\right\},\quad \left.\begin{array}{l}xy+3y-x-3=xy \\ xy-2y+x-2=xy\end{array}\right\},$$

$$\left.\begin{array}{l}3y-x=3 \\ x-2y=2\end{array}\right\}\text{ adding, } y=5, \text{ and hence } x=12.$$

54. Let x be the number that voted for A and C,

y „ „ for A and B,

z „ „ for A only.

$$\left.\begin{array}{r}\text{Then } x+\ y+\ \ z=1056 \\ x+85+\ \ 98=933 \\ y+85+744=987\end{array}\right\}, \text{ hence } x=750,\ y=158,\ z=148.$$

55. Let x be the distance in miles; y the rate in miles an hour.

$$\left.\begin{array}{l}\text{Then } \dfrac{x}{y+\frac{1}{2}}=\dfrac{x}{y}-1\frac{1}{2} \\ \text{and } \dfrac{x}{y-\frac{1}{2}}=\dfrac{x}{y}+2\frac{1}{2}\end{array}\right\}, \text{ or, } \left.\begin{array}{l}\dfrac{2x}{2y+1}=\dfrac{2x-3y}{2y} \\ \dfrac{2x}{2y-1}=\dfrac{2x+5y}{2y}\end{array}\right\}$$

$$\left.\begin{array}{l}4xy=4xy+2x-\ \ 6y^2-3y \\ 4xy=4xy-2x+10y^2-5y\end{array}\right\},\quad \left.\begin{array}{l}2x=\ \ 6y^2+3y \\ 2x=10y^2-5y\end{array}\right\}$$

hence $6y^2+3y=10y^2-5y$; $8y=4y^2$; $8=4y$; $y=2$, etc.

56. Suppose the first crew pulls x strokes of y yards each in a minute.

Then the second crew pulls $\frac{8x}{9}$ strokes of $\frac{90y}{79}$ yards each in a minute.

Hence in one minute the first crew pulls over xy yards,

and in one minute the second crew pulls over $\frac{80xy}{79}$ yards;

$\therefore$ the second is the faster crew.

Again, the second crew gains $\frac{xy}{79}$ yards in a minute;

$\therefore$ it gains $\frac{xy}{79}\div\frac{8x}{9}$ yards in a stroke, or, $\frac{9y}{79\times 8}$ yards.

Now it has to gain $\frac{4 \times 90y}{79}$ yards;

$\therefore$ it must take $\frac{4 \times 90y}{79} \div \frac{9y}{79 \times 8}$ strokes, or, 320 strokes.

57. Let x be the rate of the sculler; y the rate of the barges.

Then $\frac{b}{x}$ = time he takes to meet the first barge

$\frac{a-b}{y}$ = time the first barge takes to meet him.

Hence $\frac{b}{x} = \frac{a-b}{y}$, or, $\frac{b}{a-b} = \frac{x}{y}$. (1)

Also $\frac{b'}{x}$ = time he takes to overtake the second barge

$\frac{b'-a}{y}$ = time the second barge is in motion before he overtakes it;

$\therefore \frac{b'}{x} = \frac{b'-a}{y}$, or, $\frac{b'}{b'-a} = \frac{x}{y}$ (2)

From (1) and (2) $\frac{b}{a-b} = \frac{b'}{b'-a}$, or, $bb' - ab = ab' - bb'$,

or $2bb' = ab + ab'$, or, $\frac{2bb'}{abb'} = \frac{ab}{abb'} + \frac{ab'}{abb'}$, or, $\frac{2}{a} = \frac{1}{b'} + \frac{1}{b}$.

58. Let x be the number of feet passed over by the longer train in a second,

y the number of feet passed over by the shorter train in a second.

Then $\frac{3x}{2} + \frac{3y}{2} = 176$, and $6x - 6y = 176$.

Hence $x = \frac{220}{3}$ and $y = 44$.

Hence the longer train goes in miles per hour $\frac{220 \times 60 \times 60}{3 \times 3 \times 1760}$, or, 50.

and the shorter train goes in miles per hour $\frac{44 \times 60 \times 60}{3 \times 1760}$, or, 30.

59. Let x and y be the circumferences of the wheels in yards.

$$\text{Then } \frac{120}{x} = \frac{120}{y} + 6, \text{ and } \frac{120}{x+\frac{x}{4}} = \frac{120}{y+\frac{y}{5}} + 4;$$

$$\text{hence } \frac{20}{x} - \frac{20}{y} = 1, \text{ and } \frac{96}{x} - \frac{100}{y} = 4; \therefore x = 4, y = 5.$$

60. Let x be the number of hours he takes to go.

Then $10 - x$ is the number of hours he takes to return,

and $x = \frac{2}{3}(10 - x)$; whence $x = 4$, and $10 - x = 6$.

Again let y be the number of miles per hour he can row in still water, and z the number of miles per hour the stream flows.

Then $y + z = 5$, and $y - z = \frac{20}{6}$; whence $z = \frac{5}{6}$.

61. Let the digits commencing with the left be 1, a, b, c, d, e.

Then the number is $100000 + 10000a + 1000b + 100c + 10d + e$.

When the 1 is removed to the unit's place

the new number is $100000a + 10000b + 1000c + 100d + 10e + 1$.

Then $100000a + 10000b + 1000c + 100d + 10e + 1$

$= 300000 + 30000a + 3000b + 300c + 30d + 3e$

$\therefore\ 70000a + 7000b + 700c + 70d + 7e = 299999$

or, $10000a + 1000b + 100c + 10d + e = 42857$

$\therefore\ 100000 + 10000a + 1000b + 100c + 10d + e = 142857$

that is, the required number is 142857.

LXXX.

1.

$$\begin{array}{r|l}
 & 4a^2 + 12ab + 9b^2\ (2a + 3b \\
 & 4a^2 \\
\hline
4a + 3b & 12ab + 9b^2 \\
 & 12ab + 9b^2 \\
\hline
\end{array}$$

2.

$$\begin{array}{r|l}
 & 16k^{10} - 24k^5l^3 + 9l^6\ (4k^5 - 3l^3 \\
 & 16k^{10} \\
\hline
8k^5 - 3l^3 & -24k^5l^3 + 9l^6 \\
 & -24k^5l^3 + 9l^6 \\
\hline
\end{array}$$

3.
$$\begin{array}{l|l}
 & a^2b^2+162ab+6561\ (ab+81 \\
 & a^2b^2 \\
\hline
2ab+81 & \quad 162ab+6561 \\
 & \quad 162ab+6561 \\
\hline
\end{array}$$

4.
$$\begin{array}{l|l}
 & y^6-38y^3+361\ (y^3-19 \\
 & y^6 \\
\hline
2y^3-19 & -38y^3+361 \\
 & -38y^3+361 \\
\hline
\end{array}$$

5.
$$\begin{array}{l|l}
 & 9a^2b^2c^2-102abc+289\ (3abc-17 \\
 & 9a^2b^2c^2 \\
\hline
6abc-17 & -102abc+289 \\
 & -102abc+289 \\
\hline
\end{array}$$

6.
$$\begin{array}{l|l}
 & x^4-6x^3+19x^2-30x+25\ (x^2-3x+5 \\
 & x^4 \\
\hline
2x^2-3x & -6x^3+19x^2 \\
 & -6x^3+\ \ 9x^2 \\
\hline
2x^2-6x+5 & \qquad 10x^2-30x+25 \\
 & \qquad 10x^2-30x+25 \\
\hline
\end{array}$$

7.
$$\begin{array}{l|l}
 & 9x^4+12x^3+10x^2+4x+1\ (3x^2+2x+1 \\
 & 9x^4 \\
\hline
6x^2+2x & 12x^3+10x^2 \\
 & 12x^3+\ \ 4x^2 \\
\hline
6x^2+4x+1 & \qquad 6x^2+4x+1 \\
 & \qquad 6x^2+4x+1 \\
\hline
\end{array}$$

8.
$$\begin{array}{l|l}
 & 4r^4-12r^3+13r^2-6r+1\ (2r^2-3r+1 \\
 & 4r^4 \\
\hline
4r^2-3r & -12r^3+13r^2 \\
 & -12r^3+\ \ 9r^2 \\
\hline
4r^2-6r+1 & \qquad 4r^2-6r+1 \\
 & \qquad 4r^2-6r+1 \\
\hline
\end{array}$$

9. $4n^4+4n^3-7n^2-4n+4\ (2n^2+n-2$

$4n^4$

$4n^2+n \qquad 4n^3-7n^2$

$\qquad\qquad 4n^3+\ n^2$

$4n^2+2n-2 \qquad -8n^2-4n+4$

$\qquad\qquad -8n^2-4n+4$

10. $1\ -6x+13x^2+12x^3+4x4\ (1-3x+2x^2$

1

$2-3x \qquad -6x+13x^2$

$\qquad\qquad -6x+\ 9x^2$

$2-6x+2x^2 \qquad 4x^2+12x^3+4x^4$

$\qquad\qquad 4x^4-12x^3+4x^4$

11. $x^6-4x^5+10x^4-12x^3+9x^2\ (x^3-2x^2+3x$

$x6$

$2x^3-2x^2 \qquad -4x^5+10x^4$

$\qquad\qquad -4x^5+\ 4x^4$

$2x^3-4x^2+3x \qquad 6x^4-12x^3+9x^2$

$\qquad\qquad 6x^4-12x^3+9x^2$

12. $4y^4-12y^3z+25y^2z^2-24yz^3+16z^4\ (2y^2-3yz+4z$

$4y^4$

$4y^2-3yz \qquad -12y^3z+25y^2z^2$

$\qquad\qquad -12y^3z+9y^2z^2$

$4y^2-6yz+4z^2 \qquad 16y^2z^2-24yz^3+16z^4$

$\qquad\qquad 16y^2z^2-24yz^3+16z^4$

13. $a^2+4ab+4b^2+9c^2+6ac+12bc\,(a+2b+3c$

a^2

$2a+2b$ $\quad 4ab+4b^2$

$4ab+4b^2$

$2a+4b+3c$ $\quad 6ac+12bc+9c^2$

$6ac+12bc+9c^2$

14. $a^6+2a^5b+3a^4b^2+4a^3b^3+3a^2b^4+2ab^5+b^6\,(a^3+a^2b+ab^2+b^3$

a^6

$2a^3+a^2b$ $\quad 2a^5b+3a^4b^2$

$2a^5b+a^4b^2$

$2a^3+2a^2b+ab^2$ $\quad 2a^4b^2+4a^3b^3+3a^2b^4$

$2a^4b^2+2a^3b^3+a^2b^4$

$2a^3+2a^2b+2ab^2+b^3$ $\quad 2a^3b^3+2a^2b^4+2ab^5+b^6$

$2a^3b^3+2a^2b^4+2ab^5+b^6$

15. $x^6-4x^5+6x^3+8x^2+4x+1\,(x^3-2x^2-2x-1$

x^6

$2x^3-2x^2$ $\quad -4x^5+6x^3$

$-4x^5+4x^4$

$2x^3-4x^2-2x$ $\quad -4x^4+6x^3+8x^2$

$-4x^4+8x^3+4x^2$

$2x^3-4x^2-4x-1$ $\quad -2x^3+4x^2+4x+1$

$-2x^3+4x^2+4x+1$

16. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4\,(2x^2+2ax+4b^2$

$4x^4$

$4x^2+2ax$ $\quad 8ax^3+4a^2x^2$

$8ax^3+4a^2x^2$

$4x^2+4ax+4b^2$ $\quad 16b^2x^2+16ab^2x+16b^4$

$16b^2x^2+16ab^2x+16b^4$

17.

$$\begin{array}{ll}
 & 9-24x+58x^2-116x^3+129x^4-140x^5+100x^6\,(3-4x+7x^2-10x^3 \\
 & 9 \\
6-4x & -24x+58x^2 \\
 & -24x+16x^2 \\
6-8x+7x^2 & \quad 42x^2-116x^3+129x^4 \\
 & \quad 42x^2-56x^3+49x^4 \\
6-8x+14x^2-10x^3 & \qquad -60x^3+80x^4-140x^5+100x^6 \\
 & \qquad -60x^3+80x^4-140x^5+100x^6
\end{array}$$

18.

$$\begin{array}{ll}
 & 16a^4-40a^3b+25a^2b^2-80ab^2x+64b^2x^2+64a^2bx\,(4a^2-5ab+8bx \\
 & 16a^4 \\
8a^2-5ab & -40a^3b+25a^2b^2 \\
 & -40a^3b+25a^2b^2 \\
8a^2-10ab+8bx & \qquad 64a^2bx-80ab^2x+64b^2x^2 \\
 & \qquad 64a^2bx-80ab^2x+64b^2x^2
\end{array}$$

19.

$$\begin{array}{ll}
 & 9a^4-24a^3p^3-30a^2t+16a^2p^6+40ap^3t+25t^2\,(3a^2-4ap^3-5t \\
 & 9a^4 \\
6a^2-4ap^3 & -24a^3p^3+16a^2p^6 \\
 & -24a^3p^3+16a^2p^6 \\
6a^2-8ap^3-5t & \qquad -30a^2t+40ap^3t+25t^2 \\
 & \qquad -30a^2t+40ap^3t+25t^2
\end{array}$$

20.

$$\begin{array}{ll}
 & 4y^4x^2-12y^3x^3+17y^2x^4-12yx^5+4x^6\,(2y^2x-3yx^2+2x^3 \\
 & 4y^4x^2 \\
4y^2x-3yx^2 & -12y^3x^3+17y^2x^4 \\
 & -12y^3x^3+9y^2x^4 \\
4y^2x-6yx^2+2x^3 & \qquad 8y^2x^4-12yx^5+4x^6 \\
 & \qquad 8y^2x^4-12yx^5+4x^6
\end{array}$$

21.

$$
\begin{array}{ll}
 & 25x^4y^2 - 30x^3y^3 + 29x^2y^4 - 12xy^5 + 4y^6\,(5x^2y - 3xy^2 + 2y^3 \\
 & 25x^4y^2 \\ \hline
10x^2y - 3xy^2 & \quad -30x^3y^3 + 29x^2y^4 \\
 & \quad -30x^3y^3 + 9x^2y^2 \\ \hline
10x^2y - 6xy^2 + 2y^3 & \qquad\qquad 20x^2y^4 - 12xy^5 + 4y^6 \\
 & \qquad\qquad 20x^2y^4 - 12xy^5 + 4y^6 \\ \hline
\end{array}
$$

22.

$$
\begin{array}{ll}
 & 16x^4 - 24x^3y + 25x^2y^2 - 12xy^3 + 4y^4\,(4x^2 - 3xy + 2y^2 \\
 & 16x^4 \\ \hline
8x^2 - 3xy & \quad -24x^3y + 25x^2y^2 \\
 & \quad -24x^3y + 9x^2y^2 \\ \hline
8x^2 - 6xy + 2y^2 & \qquad\qquad 16x^2y^2 - 12xy^3 + 4y^4 \\
 & \qquad\qquad 16x^2y^2 - 12xy^3 + 4y^4 \\ \hline
\end{array}
$$

23.

$$
\begin{array}{ll}
 & 9a^2 - 12ab + 24ac - 16bc + 4b^2 + 16c^2\,(3a - 2b + 4c \\
 & 9a^2 \\ \hline
6a - 2b & \quad -12ab + 4b^2 \\
 & \quad -12ab + 4b^2 \\ \hline
6a - 4b + 4c & \qquad\qquad 24ac - 16bc + 16c^2 \\
 & \qquad\qquad 24ac - 16bc + 16c^2 \\ \hline
\end{array}
$$

24.

$$
\begin{array}{ll}
 & x^4 + 19x^2 + 25 - 6x^3 - 30x\,(x^2 - 3x + 5 \\
 & x^4 \\ \hline
2x^2 - 3x & \quad -6x^3 + 19x^2 \\
 & \quad -6x^3 + 9x^2 \\ \hline
2x^2 - 6x + 5 & \qquad\qquad 10x^2 - 30x + 25 \\
 & \qquad\qquad 10x^2 - 30x + 25 \\ \hline
\end{array}
$$

25. $25x^2-20xy+4y^2+9z^2-12yz+30xz\,(5x-2y+3z$

$25x^2$

$10x-2y$	$-20xy+4y^2$
	$-20xy+4y^2$
$10x-4y+3z$	$30xz-12yz+9z^2$
	$30xz-12yz+9z^2$

26. $4x^4-4x^2y+4x^2y^2+y^4-2y^3+y^2\,(2x^2-y+y^2$

$4x^4$

$4x^2-y$	$-4x^2y+y^2$
	$-4x^2y+y^2$
$4x^2-2y+y^2$	$4x^2y^2-2y^3+y^4$
	$4x^2y^2-2y^3+y^4$

LXXXI.

1. $4a^6+\frac{a^2b^4}{16}-a^4b^2\,(2a^3-\frac{ab^2}{4}$

$4a^6$

$4a^3-\frac{ab^2}{4}$	$-a^4b^2+\frac{a^2b^4}{16}$
	$-a^4b^2+\frac{a^2b^4}{16}$

2. $\frac{9}{a^2}-2+\frac{a^2}{9}\left(\frac{3}{a}-\frac{a}{3}\right.$

$\frac{9}{a^2}$

$\frac{6}{a}-\frac{a}{3}$	$-2+\frac{a^2}{9}$
	$-2+\frac{a^2}{9}$

3. $$a^4 - 2 + \frac{1}{a^4}\left(a^2 - \frac{1}{a^2}\right.$$

$$\begin{array}{l|l} & a^4 \\ \hline 2a^2 - \frac{1}{a^2} & -2 + \frac{1}{a^4} \\ & -2 + \frac{1}{a^4} \\ \hline \end{array}$$

4. $$\frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}\left(\frac{a}{b} + \frac{b}{a}\right.$$

$$\begin{array}{l|l} & \frac{a^2}{b^2} \\ \hline \frac{2a}{b} + \frac{b}{a} & 2 + \frac{b^2}{a^2} \\ & 2 + \frac{b^2}{a^2} \\ \hline \end{array}$$

5. $$x^4 - 2x^3 + 2x^2 - x + \frac{1}{4}\ \left(x^2 - x + \frac{1}{2}\right.$$

$$\begin{array}{l|l} & x^4 \\ \hline 2x^2 - x & -2x^3 + 2x^2 \\ & -2x^3 + x^2 \\ \hline 2x^2 - 2x + \frac{1}{2} & x^2 - x + \frac{1}{4} \\ & x^2 - x + \frac{1}{4} \\ \hline \end{array}$$

6. $$x^4 + 2x^3 - x + \frac{1}{4}\ \left(x^2 + x - \frac{1}{2}\right.$$

$$\begin{array}{l|l} & x^4 \\ \hline 2x^2 + x & 2x^3 - x \\ & 2x^3 + x^2 \\ \hline 2x^2 + 2x - \frac{1}{2} & -x^2 - x + \frac{1}{4} \\ & -x^2 - x + \frac{1}{4} \\ \hline \end{array}$$

7. $$4a^2 - 12ab + ab^2 + 9b^2 - \frac{3b^3}{2} + \frac{b^4}{16}\left(2a - 3b + \frac{b^2}{4}\right.$$

$$\begin{array}{l|l}
 & 4a^2 \\
\hline
4a - 3b & -12ab + ab^2 + 9b^2 \\
 & -12ab + 9b^2 \\
\hline
4a - 6b + \frac{b^2}{4} & \quad ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\
 & \quad ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\
\hline
\end{array}$$

8. $$x^4 + 8x^2 + 24 + \frac{16}{x^4} + \frac{32}{x^2}\ \left(x^2 + 4 + \frac{4}{x^2}\right.$$

$$\begin{array}{l|l}
 & x^4 \\
\hline
2x^2 + 4 & 8x^2 + 24 \\
 & 8x^2 + 16 \\
\hline
2x^2 + 8 + \frac{4}{x^2} & \quad 8 + \frac{32}{x^2} + \frac{16}{x^4} \\
 & \quad 8 + \frac{32}{x^2} + \frac{16}{x^4} \\
\hline
\end{array}$$

9. $$\frac{16}{9}a^6x^2 + \frac{16}{3}a^5x + 4a^4 - 2a^3x - 3a^2 + \frac{9}{16}\left(\frac{4a^3x}{3} + 2a^2 - \frac{3}{4}\right.$$

$$\begin{array}{l|l}
 & \frac{16}{9}a^6x^2 \\
\hline
\frac{8a^3x}{3} + 2a^2 & \frac{16}{3}a^5x + 4a^4 \\
 & \frac{16}{3}a^5x + 4a^4 \\
\hline
\frac{8a^3x}{3} + 4a^2 - \frac{3}{4} & \quad -2a^3x - 3a^2 + \frac{9}{16} \\
 & \quad -2a^3x - 3a^2 + \frac{9}{16} \\
\hline
\end{array}$$

10.
$$\frac{1}{x^2}-\frac{4}{xy}+\frac{4}{y^2}+\frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2}\Big(\frac{1}{x}-\frac{2}{y}+\frac{3}{z}$$

$$\begin{array}{l|l}
 & \frac{1}{x^2} \\
\hline
\frac{2}{x}-\frac{2}{y} & -\frac{4}{xy}+\frac{4}{y^2} \\
 & -\frac{4}{xy}+\frac{4}{y^2} \\
\hline
\frac{2}{x}-\frac{4}{y}+\frac{3}{z} & \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2} \\
 & \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2} \\
\hline
\end{array}$$

11.
$$36m^2-\frac{48m}{n}+\frac{16}{n^2}+\frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25}\Big(6m-\frac{4}{n}+\frac{p}{5}$$

$$\begin{array}{l|l}
 & 36m^2 \\
\hline
12m-\frac{4}{n} & -\frac{48m}{n}+\frac{16}{n^2} \\
 & -\frac{48m}{n}+\frac{16}{n^2} \\
\hline
12m-\frac{8}{n}+\frac{p}{5} & \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25} \\
 & \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25} \\
\hline
\end{array}$$

12.
$$a^2b^2-6abcd+9c^2d^2+\frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49}\ (ab-3cd+\frac{ef}{7}$$

$$\begin{array}{l|l}
 & a^2b^2 \\
\hline
2ab-3cd & -6abcd+9c^2d^2 \\
 & -6abcd+9c^2d^2 \\
\hline
2ab-6cd+\frac{ef}{7} & \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49} \\
 & \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49} \\
\hline
\end{array}$$

13.

$$\frac{4x^2}{z^2}-\frac{12xy}{z^2}+\frac{9y^2}{z^2}+4-\frac{6y}{x}+\frac{z^2}{x^2}\Big(\frac{2x}{z}-\frac{3y}{z}+\frac{z}{x}$$

$$\frac{4x^2}{z^2}$$

$$\frac{4x}{z}-\frac{3y}{z}\quad\Big\backslash\quad -\frac{12xy}{z^2}+\frac{9y^2}{z^2}$$

$$-\frac{12xy}{z^2}+\frac{9y^2}{z^2}$$

$$\frac{4x}{z}-\frac{3y}{z}+\frac{z}{x}\quad\Big\backslash\quad 4-\frac{6y}{x}+\frac{z^2}{x^2}$$

$$4-\frac{6y}{x}+\frac{z^2}{x^2}$$

14.

$$\frac{4m^2}{n^2}-\frac{16m}{n}+4+\frac{24n}{m}+\frac{9n^2}{m^2}\Big(\frac{2m}{n}-4-\frac{3n}{m}$$

$$\frac{4m^2}{n^2}$$

$$\frac{4m}{n}-4\quad\Big\backslash\quad -\frac{16m}{n}+4$$

$$-\frac{16m}{n}+16$$

$$\frac{4m}{n}-8-\frac{3n}{m}\quad\Big\backslash\quad -12+\frac{24n}{m}+\frac{9n^2}{m^2}$$

$$-12+\frac{24n}{m}+\frac{9n^2}{m^2}$$

15.

$$\frac{a^2}{9}-\frac{ab}{6}+\frac{b^2}{16}+\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}\left(\frac{a}{3}-\frac{b}{4}+\frac{c}{5}-\frac{d}{2}\right.$$

$$\frac{a^2}{9}$$

$$\frac{2a}{3}-\frac{b}{4} \qquad -\frac{ab}{6}+\frac{b^2}{16}$$

$$-\frac{ab}{6}+\frac{b^2}{16}$$

$$\frac{2a}{3}-\frac{b}{2}+\frac{c}{5} \qquad \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}$$

$$\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}$$

$$\frac{2a}{3}-\frac{b}{2}+\frac{2c}{5}-\frac{d}{2} \qquad -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}$$

$$-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}$$

16.

$$49x^4-28x^3-17x^2+6x+\frac{9}{4}\left(7x^2-2x-\frac{3}{2}\right.$$

$$49x^4$$

$$14x^2-2x \qquad -28x^3-17x^2$$

$$-28x^3+4x^2$$

$$14x^2-4x-\frac{3}{2} \qquad -21x^2+6x+\frac{9}{4}$$

$$-21x^2+6x+\frac{9}{4}$$

17.

$$9x^4-3ax^3+6bx^3+\frac{a^2x^2}{4}-abx^2+b^2x^2\left(3x^2-\frac{ax}{2}+bx\right.$$

$$9x^4$$

$$6x^2-\frac{ax}{2} \qquad -3ax^3+\frac{a^2x^2}{4}$$

$$-3ax^3+\frac{a^2x^2}{4}$$

$$6x^2-ax+bx \qquad 6bx^3-abx^2+b^2x^2$$

$$6bx^3-abx^2+b^2x^2$$

18.

$$9x^4 - 2x^3 - \frac{161}{9}x^2 + 2x + 9 \left(3x^2 - \frac{x}{3} - 3\right.$$

$$9x^4$$

$$6x^2 - \frac{x}{3} \quad \Big| \quad -2x^3 - \frac{161}{9}x^2$$

$$-2x^3 + \frac{x^2}{9}$$

$$6x^2 - \frac{2x}{3} - 3 \quad \Big| \quad -18x^2 + 2x + 9$$

$$-18x^2 + 2x + 9$$

LXXXIII.

1.

$$a^3 - 3a^2b + 3ab^2 - b^3 (a - b$$

$$a^3$$

$$3a^2 - 3ab + b^2 \quad \Big| \quad -3a^2b + 3ab^2 - b^3$$

$$-3a^2b + 3ab^2 - b^3$$

2.

$$8a^3 + 12a^2 + 6a + 1 (2a + 1$$

$$8a^3$$

$$12a^2 + 6a + 1 \quad \Big| \quad 12a^2 + 6a + 1$$

$$12a^2 + 6a + 1$$

3.

$$a^3 + 24a^2b + 192ab^2 + 512b^3 (a + 8b$$

$$a^3$$

$$3a^2 + 24ab + 64b^2 \quad \Big| \quad 24a^2b + 192ab^2 + 512b^3$$

$$24a^2b + 192ab^2 + 512b^3$$

4.

$$a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 (a + b + c$$

$$a^3$$

$$3a^2 + 3ab + b^2 \quad \Big| \quad 3a^2b + 3ab^2 + b^3$$

$$3a^2b + 3ab^2 + b^3$$

$$3a^2 + 6ab + 3b^2 + 3ac + 3bc + c^2 \quad \Big| \quad 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$$

$$3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$$

5.

$$x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3 (x - y + z$$

$$x^3$$

$$3x^2 - 3xy + y^2 \quad \Big| \quad -3x^2y + 3xy^2 - y^3$$

$$-3x^2y + 3xy^2 - y^3$$

$$3x^2 - 6xy + 3y^2 + 3xz - 3yz + z^2 \quad \Big| \quad 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$$

$$3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$$

6.

$$
\begin{array}{ll}
 & 27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1\,(3x^2 - 2x + 1 \\
 & 27x^6 \\
\hline
27x^4 - 18x^3 + 4x^2 & -54x^5 + 63x^4 - 44x^3 \\
 & -54x^5 + 36x^4 - 8x^3 \\
\hline
27x^4 - 36x^3 + 21x^2 - 6x + 1 & 27x^4 - 36x^3 + 21x^2 - 6x + 1 \\
 & 27x^4 - 36x^3 + 21x^2 - 6x + 1 \\
\hline
\end{array}
$$

7.

$$
\begin{array}{ll}
 & 1 - 3a + 6a^2 - 7a^3 + 6a^4 - 3a^5 + a^6\,(1 - a + a^2 \\
 & 1 \\
\hline
3 - 3a + a^2 & -3a + 6a^2 - 7a^3 \\
 & -3a + 3a^2 - a^3 \\
\hline
3 - 6a + 6a^2 - 3a^3 + a^4 & 3a^2 - 6a^3 + 6a^4 - 3a^5 + a^6 \\
 & 3a^2 - 6a^3 + 6a^4 - 3a^5 + a^6 \\
\hline
\end{array}
$$

8.

$$
\begin{array}{ll}
 & x^3 - 3x^2y + 3xy^2 - y^3 + 8z^3 + 6x^2z - 12xyz + 6y^2z + 12xz^2 - 12yz^2 \\
 & x^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x - y + 2z \\
\hline
3x^2 - 3xy + y^2 & -3x^2y + 3xy^2 - y^3 \\
 & -3x^2y + 3xy^2 - y^3 \\
\hline
3x^2 - 6xy + 3y^2 & 6x^2z - 12xyz + 6y^2z + 12xz^2 - 12yz^2 + 8z^3 \\
\quad + 6xz - 6yz + 4z^2 & 6x^2z - 12xyz + 6y^2z + 12xz^2 - 12yz^2 + 8z^3 \\
\hline
\end{array}
$$

9.

$$
\begin{array}{ll}
 & a^6 - 12a^5 + 54a^4 - 112a^3 + 108a^2 - 48a + 8\,(a^2 - 4a + 2 \\
 & a^6 \\
\hline
3a^4 - 12a^3 + 16a^2 & -12a^5 + 54a^4 - 112a^3 \\
 & -12a^5 + 48a^4 - 64a^3 \\
\hline
3a^4 - 24a^3 + 54a^2 - 24a + 8 & 6a^4 - 48a^3 + 108a^2 - 48a + 8 \\
 & 6a^4 - 48a^3 + 108a^2 - 48a + 8 \\
\hline
\end{array}
$$

10.

$$8m^6 - 36m^5 + 66m^4 - 63m^3 + 33m^2 - 9m + 1\,(2m^2 - 3m + 1$$
$$8m^6$$

$12m^4 - 18m^3 + 9m^2$

$$-36m^5 + 66m^4 - 63m^3$$
$$-36m^5 + 54m^4 - 27m^3$$

$12m^4 - 36m^3 + 33m^2 - 9m + 1$

$$12m^4 - 36m^3 + 33m^2 - 9m + 1$$
$$12m^4 - 36m^3 + 33m^2 - 9m + 1$$

11.

$$x^3 + 6x^2y + 12xy^2 + 8y^3 - 3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3\,(x + 2y - z$$
$$x^3$$

$3x^2 + 6xy + 4y^2$

$$6x^2y + 12xy^2 + 8y^3$$
$$6x^2y + 12xy^2 + 8y^3$$

$3x^2 + 12xy + 12y^2 - 3xz - 6yz + z^2$

$$-3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$$
$$-3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$$

12.

$$8m^3 - 36m^2n + 54mn^2 - 27n^3 - 12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3\,(2m - 3n - r$$
$$8m^3$$

$12m^2 - 18mn + 9n^2$

$$-36m^2n + 54mn^2 - 27n^3$$
$$-36m^2n + 54mn^2 - 27n^3$$

$12m^2 - 36mn + 27n^2 - 6mr - 9nr + r^2$

$$-12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3$$
$$-12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3$$

13.

$$m^3 + 3m^2 - 5 + \frac{3}{m^2} - \frac{1}{m^3}\,\left(m + 1 - \frac{1}{m}\right.$$
$$m^3$$

$3m^2 + 3m + 1$

$$3m^2 - 5 + \frac{3}{m^2}$$
$$3m^2 + 3m + 1$$

$3m^2 + 6m - \frac{3}{m} + \frac{1}{m^2}$

$$-3m - 6 + \frac{3}{m^2} - \frac{1}{m^3}$$
$$-3m - 6 + \frac{3}{m^2} - \frac{1}{m^3}$$

LXXXIV.

1.

$$\begin{array}{l|l}
 & 16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4\,(4a^2 - 12ax + 9x^2 \\
 & 16a^4 \\ \hline
8a^2 - 12ax & -96a^3x + 216a^2x^2 \\
 & -96a^3x + 144a^2x^2 \\ \hline
8a^2 - 24ax + 9x^2 & 72a^2x^2 - 216ax^3 + 81x^4 \\
 & 72a^2x^2 - 216ax^3 + 81x^4 \\ \hline
\end{array}$$

and the square root of $4a^2 - 12ax + 9x^2$ is $2a - 3x$.

2.

$$\begin{array}{l|l}
 & 1 - 8a + 24a^2 - 32a^3 + 16a^4\,(1 - 4a + 4a^2 \\
 & 1 \\ \hline
2 - 4a & -8a + 24a^2 \\
 & -8a + 16a^2 \\ \hline
2 - 8a + 4a^2 & 8a^2 - 32a^3 + 16a^4 \\
 & 8a^2 - 32a^3 + 16a^4 \\ \hline
\end{array}$$

and the square root of $1 - 4a + 4a^2$ is $1 - 2a$.

3.

$$\begin{array}{l|l}
 & 625 + 2000x + 2400x^2 + 1280x^3 + 256x^4\,(25 + 40x + 16x^2 \\
 & 625 \\ \hline
50 + 40x & 2000x + 2400x^2 \\
 & 2000x + 1600x^2 \\ \hline
50 + 80x + 16x^2 & 800x^2 + 1280x^3 + 256x^4 \\
 & 800x^2 + 1280x^3 + 256x^4 \\ \hline
\end{array}$$

and the square root of $25 + 40x + 16x^2$ is $5 + 4x$.

4. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \,(a^3 - 3a^2b + 3ab^2 - b^3$

a^6

$2a^3 - 3a^2b$ | $-6a^5b + 15a^4b^2$

$-6a^5b + 9a^4b^2$

$2a^3 - 6a^2b + 3ab^2$ | $6a^4b^2 - 20a^3b^3 + 15a^2b^4$

$6a^4b^2 - 18a^3b^3 + 9a^2b^4$

$2a^3 - 6a^2b + 6ab^2 - b^3$ | $-2a^3b^3 + 6a^2b^4 - 6ab^5 + b^6$

$-2a^3b^3 + 6a^2b^4 - 6ab^5 + b^6$

and the cube root of $a^3 - 3a^2b + 3ab^2 - b^3$ is $a - b$.

5. $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 \,(x^3 + 3x^2 + 3x + 1$

x^6

$2x^3 + 3x^2$ | $6x^5 + 15x^4$

$6x^5 + 9x^4$

$2x^3 + 6x^2 + 3x$ | $6x^4 + 20x^3 + 15x^2$

$6x^4 + 18x^3 + 9x^2$

$2x^3 + 6x^2 + 6x + 1$ | $2x^3 + 6x^2 + 6x + 1$

$2x^3 + 6x^2 + 6x + 1$

and the cube root of $x^3 + 3x^2 + 3x + 1$ is $x + 1$.

6. $m^6 - 12m^5 + 60m^4 - 160m^3 + 240m^2 - 192m + 64 \,(m^3 - 6m^2 + 12m - 8$

m^6

$2m^3 - 6m^2$ | $-12m^5 + 60m^4$

$-12m^5 + 36m^4$

$2m^3 - 12m^2 + 12m$ | $24m^4 - 160m^3 + 240m^2$

$24m^4 - 144m^3 + 144m^2$

$2m^3 - 12m^2 + 24m - 8$ | $-16m^3 + 96m^2 - 192m + 64$

$-16m^3 + 96m^2 - 192m + 64$

and the cube root of $m^3 - 6m^2 + 12m - 8$ is $m - 2$.

LXXXV.

1. $x=\pm 8$. 2. $x=\pm ab$. 3. $x^2=10000$; $x=\pm 100$.

4. $x^2=49$; $x=\pm 7$. 5. $3x^2=33$; $x^2=11$; $x=\pm\sqrt{11}$.

6. $x^2=64a^4c^6$, etc. 7. $4x^2-48=3x^2-12$; $x^2=36$, etc.

8. $250000-x^2=233359$; $x^2=16641$; $x=\pm 129$.

9. $8112=3x^2$; $x^2=2704$, etc.

10. $\frac{11x^2}{2}-18x+65=9x^2-18x+9$; $11x^2-18x^2=-112$; $7x^2=112$, etc.

11. $mx^2=q-n$; $x^2=\frac{q-n}{m}$, etc.

12. $x^2-ax+b=ax^2-ax$; $ax^2-x^2=b$; $x^2=\frac{b}{a-1}$, etc.

13. $180x^2-225=114x^2+171$; $66x^2=396$; $x^2=6$, etc.

14. $42x^2-126=35x^2-70$; $7x^2=56$; $x^2=8$; $x^2=4\times 2$; $x=\pm 2\sqrt{2}$.

LXXXVI.

1. $x^2+6x+9=81$; $x+3=\pm 9$, etc.

2. $x^2+12x+36=100$; $x+6=\pm 10$, etc.

3. $x^2+14x+49=64$; $x+7=\pm 8$, etc.

4. $x^2+46x+529=625$; $x+23=\pm 25$, etc.

5. $x^2+128x+4096=4489$; $x+64=\pm 67$, etc.

6. $x^2+8x+16=81$; $x+4=\pm 9$, etc.

7. $x^2+18x+81=324$; $x+9=\pm 18$, etc.

8. $x^2+16x+64=484$; $x+8=\pm 22$, etc.

LXXXVII.

1. $x^2-6x+9=16$; $x-3=\pm 4$, etc.

2. $x^2-4x+4=9$; $x-2=\pm 3$, etc.

3. $x^2-20x+100=121$; $x-10=\pm 11$, etc.

4. $x^2-2x+1=64$; $x-1=\pm 8$, etc.

5. $x^2-12x+36=4$; $x-6=\pm 2$, etc.

6. $x^2-14x+49=4$; $x-7=\pm 2$, etc.

7. $x^2-234x+13689=1$; $x-117=\pm 1$, etc.

8. $x^2-5x+6=15x+42$; $x^2-20x=36$; $x^2-20x+100=136$;
   $x-10=\pm\sqrt{4\times 34}$; $x=10\pm 2\sqrt{34}$.

9. $3x^2-17x-2x^2-5x+120=0$; $x^2-22x+121=1$, etc.

10. $x^2-10x+25+x^2-14x+49=x^2-8x+46$; $x^2-16x=-28$, etc.

LXXXVIII.

1. $x^2+7x+\frac{49}{4}=\frac{169}{4}$; $x+\frac{7}{2}=\pm\frac{13}{2}$, etc.

2. $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

3. $x^2+9x+\frac{81}{4}=\frac{256}{4}$; $x+\frac{9}{2}=\pm\frac{16}{2}$, etc.

4. $x^2-13x+\frac{169}{4}=\frac{729}{4}$; $x-\frac{13}{2}=\pm\frac{27}{2}$, etc.

5. $x^2+x+\frac{1}{4}=\frac{9}{16}$; $x+\frac{1}{2}=\pm\frac{3}{4}$, etc.

6. $x^2-x+\frac{1}{4}=\frac{289}{4}$; $x-\frac{1}{2}=\pm\frac{17}{2}$, etc.

7. $x^2+37x+\frac{1369}{4}=\frac{16129}{4}$; $x+\frac{37}{2}=\pm\frac{127}{2}$, etc.

8. $x^2-x+\frac{1}{4}=\frac{225}{4}$; $x-\frac{1}{2}=\pm\frac{15}{2}$, etc.

9. $5x-x^2+2x^2-14x-10x+60=0$; $x^2-19x=-60$;

$x^2-19x+\frac{361}{4}=\frac{121}{4}$, etc.

10. $35x^2-312x+693-34x^2+21x+45=448$; $x^2-291x=-290$;

$x^2-291x+\frac{84681}{4}=\frac{83521}{4}$; $x-\frac{291}{2}=\pm\frac{289}{2}$, etc.

LXXXIX.

1. $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{36}{9}$; $x-\frac{1}{3}=\pm\frac{6}{3}$, etc.

2. $x^2+\frac{4}{5}x+\frac{4}{25}=\frac{1}{25}$; $x+\frac{2}{5}=\pm\frac{1}{5}$, etc.

3. $x^2-\frac{28x}{9}+\frac{196}{81}=\frac{169}{81}$; $x-\frac{14}{9}=\pm\frac{13}{9}$, etc.

4. $x^2-\frac{8}{11}x+\frac{16}{121}=\frac{49}{121}$; $x-\frac{4}{11}=\pm\frac{7}{11}$, etc.

5. $x^2+\frac{4}{35}x+\frac{4}{1225}=\frac{529}{1225}$; $x+\frac{2}{35}=\pm\frac{23}{35}$, etc.

6. $x^2-\frac{16}{5}x+\frac{64}{25}=\frac{144}{25}$; $x-\frac{8}{5}=\pm\frac{12}{5}$, etc.

7. $x^2-\frac{26}{3}x+\frac{169}{9}=\frac{121}{9}$; $x-\frac{13}{3}=\pm\frac{11}{3}$, etc.

8. $x^2-\frac{4}{7}x+\frac{4}{49}=\frac{2209}{49}$; $x-\frac{2}{7}=\pm\frac{47}{7}$, etc.

XC.

1. $x^2-\frac{1}{3}x+\frac{1}{36}=\frac{289}{36}$; $x-\frac{1}{6}=\pm\frac{17}{6}$, etc.

2. $x^2-\frac{1}{5}x+\frac{1}{100}=\frac{9801}{100}$; $x-\frac{1}{10}=\pm\frac{99}{10}$, etc.

3. $x^2+\frac{1}{2x}+\frac{1}{16}=\frac{625}{16}$; $x+\frac{1}{4}=\pm\frac{25}{4}$, etc.

4. $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{1225}{16}$; $x+\frac{3}{4}=\pm\frac{35}{4}$, etc.

5. $x^2-\frac{9}{5}x+\frac{81}{100}=\frac{1681}{100}$; $x-\frac{9}{10}=\pm\frac{41}{10}$, etc.

6. $x^2-\frac{11}{2}x+\frac{121}{16}=\frac{25}{16}$; $x-\frac{11}{4}=\pm\frac{5}{4}$, etc.

7. $x^2-\frac{15}{4}x+\frac{225}{64}=\frac{2401}{64}$; $x-\frac{15}{8}=\pm\frac{49}{8}$, etc.

8. $x^2-\frac{23}{7}x+\frac{529}{196}=\frac{676}{196}$; $x-\frac{23}{14}=\pm\frac{26}{14}$, etc.

XCI.

1. $x^2+2ax+a^2=2a^2$; $x+a=\pm\sqrt{2}.a$, etc.

2. $x^2-4ax+4a^2=11a^2$; $x-2a=\pm\sqrt{11}.a$, etc.

3. $x^2+3mx+\frac{9m^2}{4}=4m^2$; $x+\frac{3m}{2}=\pm 2m$, etc.

4. $x^2-\frac{5nx}{2}+\frac{25n^2}{16}=\frac{49n^2}{16}$; $x-\frac{5n}{4}=\pm\frac{7n}{4}$, etc.

5. $x^2+(a-1)x+\frac{(a-1)^2}{4}=\frac{a^2+2a+1}{4}$; $x+\frac{a-1}{2}=\pm\frac{a+1}{2}$, etc.

6. $x^2+(a-b)x+\frac{(a-b)^2}{4}=\frac{(a+b)^2}{4}$; $x+\frac{a-b}{2}=\pm\frac{a+b}{2}$, etc.

7. $\frac{a^2}{(x+a)^2}=\frac{b^2}{(x-a)^2}$; and, taking the square root of each side,

$\frac{a}{x+a}=\pm\frac{b}{x-a}$, etc.

8. $acx^2-adx+bcx=bd$; $x^2+\frac{bc-ad}{ac}x=\frac{bd}{ac}$;

$x^2+\frac{bc-ad}{ac}x+\frac{(bc-ad)^2}{4a^2c^2}=\frac{(bc+ad)^2}{4a^2c^2}$;

$x+\frac{bc-ad}{2ac}=\pm\frac{bc+ad}{2ac}$; $x=\frac{2ad}{ac}$ or $-\frac{2bc}{2ac}$, etc.

9. $(a+b)x^2-cx=\frac{ac}{a+b}$; $x^2-\frac{cx}{a+b}=\frac{ac}{(a+b)^2}$;

$x^2-\frac{cx}{a+b}+\frac{c^2}{4(a+b)^2}=\frac{c^2+4ac}{4(a+b)^2}$; $x-\frac{c}{2(a+b)}=\pm\frac{\sqrt{c^2+4ac}}{2(a+b)}$, etc.

10. $x^2-\frac{2b^2x}{ac}=-\frac{b^4}{a^2c^2}$; $x^2-\frac{2b^2x}{ac}+\frac{b^4}{a^2c^2}=0$; $x-\frac{b^2}{ac}=0$; $x=\frac{b^2}{ac}$.

11. $x^2+\frac{3a^2+b^2}{abc}x=\frac{6a^2+ab-2b^2}{abc^2}$;

$x^2+\frac{3a^2+b^2}{abc}x+\frac{(3a^2+b^2)^2}{4a^2b^2c^2}=\frac{24a^3b+4a^2b^2-8ab^3+9a^4+6a^2b^2+b^4}{4a^2b^2c^2}$

$x+\frac{3a^2+b^2}{2abc}=\pm\frac{3a^2+4ab-b^2}{2abc}$, etc.

12. $x^2+\frac{4a^2c^2+4abd^2}{4a^2-9cd^2}x=-\frac{a^2c^4+2abc^2d^2+b^2d^4}{4a^2-9cd^2}$;

$$x^2+\frac{4a^2c^2+4abd^2}{4a^2-9cd^2}x+\frac{(2a^2c^2+2abd^2)^2}{(4a^2-9cd^2)^2}=\frac{9cd^2(a^2c^4+2abc^2d^2+b^2d^4)}{(4a^2-9cd^2)^2};$$

$$x+\frac{2a^2c^2+2abd^2}{4a^2-9cd^2}=\pm\frac{3d\sqrt{c}.(ac^2+bd^2)}{4a^2-9cd^2};$$

$$x=\frac{-2a(ac^2+bd^2)\pm 3d\sqrt{c}.(ac^2+bd^2)}{4a^2-9cd^2};$$

$$x=\frac{(-2a\pm 3d\sqrt{c})(ac^2+bd^2)}{(2a+3d\sqrt{c})(2a-3d\sqrt{c})}\text{, etc.}$$

XCII.

1. $x^2-7x=8$; $x^2-7x+\frac{49}{4}=\frac{81}{4}$; $x-\frac{7}{2}=\pm\frac{9}{2}$, etc.

2. $x^2-5x=6$; $x^2-5x+\frac{25}{4}=\frac{49}{4}$; $x-\frac{5}{2}=\pm\frac{7}{2}$, etc.

3. $x^2-11x=12$; $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

4. $x^2-13x=14$; $x^2-13x+\frac{169}{4}=\frac{225}{4}$; $x-\frac{13}{2}=\pm\frac{15}{2}$, etc.

5. $x^2+7x=18$; $x^2+7x+\frac{49}{4}=\frac{121}{4}$; $x+\frac{7}{2}=\pm\frac{11}{2}$, etc.

6. $4x^2-12x-12+x=22x-66$; $4x^2-33x=-54$; $x^2-\frac{33x}{4}=-\frac{54}{4}$;

$$x^2-\frac{33x}{4}+\frac{1089}{64}=\frac{225}{64};\ x-\frac{33}{8}=\pm\frac{15}{8}\text{, etc.}$$

7. $x^2-9x+\frac{81}{4}=\frac{1}{4}$; $x-\frac{9}{2}=\pm\frac{1}{2}$, etc.

8. $10x^2-30x-6x+6=7x^2-27x+18$; $3x^2-9x=12$; $x^2-3x=4$, etc.

9. $x^2 - [illegible]$ [illegible]

10. [illegible]

[illegible]

11. [illegible]

[illegible]

12. $x^2 - [illegible]$; $x - [illegible] = \pm [illegible]$, etc.

13. $x^2 - [illegible]$; $x - [illegible] = \pm [illegible]$, etc.

14. [illegible]; $x^2 - \frac{[illegible]}{3} = [illegible]$; [illegible], etc.

15. $2x^2 - [illegible]$; $x^2 - \frac{[illegible]}{3} = \frac{[illegible]}{3}$; [illegible], etc.

16. $x^2 - [illegible] = -[illegible]$; $x^2 - [illegible] + \frac{[illegible]}{4} = \frac{1}{4}$; $x - \frac{9}{2} = \pm\frac{1}{2}$, etc.

17. $6x^2 - [illegible]$; $x^2 - [illegible]$, etc.

18. $x^2 - [illegible]$; $x - [illegible] = \pm [illegible]$, etc.

19. [illegible]

$x^2 + \frac{15x}{54} = \frac{[illegible]}{54}$; $x^2 + \frac{15x}{54} + \frac{225}{11664} = \frac{[illegible]}{11664}$; $x + \frac{15}{108} = \pm\frac{[illegible]}{108}$, etc.

20. $7x^2 + [illegible] = 6x^2 + 16x - 70$; $7x^2 - 39x = 70$;

$x^2 - \frac{39x}{7} = 10$; $x^2 - \frac{39x}{7} + \frac{1521}{196} = \frac{3481}{196}$; $x - \frac{39}{14} = \pm\frac{59}{14}$, etc.

21. $8x^2 - [illegible] + 6x^2 + 16x - 70 = 7x^2 + 35x$; $7x^2 - 39x = 70$, etc.

22. $x^2 - 6x + 9 + 4x = 44$; $x^2 - 2x = 35$, etc.

23. $x^2+11x=7x^2-9-4x$; $6x^2-15x=9$; $2x^2-5x=3$; $x^2-\frac{5x}{2}=\frac{3}{2}$;

$x^2-\frac{5x}{2}+\frac{25}{16}=\frac{49}{16}$; $x-\frac{5}{4}=\pm\frac{7}{4}$, etc.

24. $x^2+\frac{x}{6}=\frac{2}{6}$; $x^2+\frac{x}{6}+\frac{1}{144}=\frac{49}{144}$; $x+\frac{1}{12}=\pm\frac{7}{12}$, etc.

25. $x^2-\frac{x}{2}+\frac{1}{16}=\frac{25}{144}$; $x-\frac{1}{4}=\pm\frac{5}{12}$, etc.

26. $x^2-x+\frac{1}{4}=\frac{841}{4}$; $x-\frac{1}{2}=\pm\frac{29}{2}$, etc.

27. $6x+2x+2=3x^2+3x$; $3x^2-5x=2$; $x^2-\frac{5x}{3}=\frac{2}{3}$, etc.

28. $4x^2-33=x$; $x^2-\frac{x}{4}=\frac{33}{4}$; $x^2-\frac{x}{4}+\frac{1}{64}=\frac{529}{64}$, etc.

29. $2x^2=3x^2-3x+2x^2-4x+2$; $x^2-\frac{7x}{3}=-\frac{2}{3}$, etc.

30. $x^2-\frac{7x}{15}=\frac{46}{15}$; $x^2-\frac{7x}{15}+\frac{49}{900}=\frac{2809}{900}$; $x-\frac{7}{30}=\pm\frac{53}{30}$, etc.

31. $5x+10-10x+20=3x^2-12$; $x^2+\frac{5x}{3}=14$, etc.

32. $4x^2-100+40x-4x^2=75x-15x^2$; $15x^2-35x=100$;

$x^2-\frac{7x}{3}=\frac{20}{3}$; $x^2-\frac{7x}{3}+\frac{49}{36}=\frac{289}{36}$, etc.

33. $90x-126+18x=22x^2$; $x^2-\frac{54x}{11}=-\frac{63}{11}$; $x^2-\frac{54x}{11}+\frac{729}{121}=\frac{36}{121}$, etc.

34. $3x^2-5x=7x+420$; $x^2-4x=140$, etc.

35. $\frac{48-12x+40-8x}{20-9x+x^2}=\frac{32}{x+2}$; $\frac{22-5x}{20-9x+x^2}=\frac{8}{x+2}$;

$44+12x-5x^2=160-72x+8x^2$; $13x^2-84x=-116$;

$x^2-\frac{84x}{13}=-\frac{116}{13}$; $x^2-\frac{84x}{13}+\frac{1764}{169}=\frac{256}{169}$, etc.

36. $10x^2+490-140x+10x^2=203x-29x^2$; $49x^2-343x=-490$;
$x^2-7x=-10$, etc.

37. $x^2+(a+b)x+\dfrac{(a+b)^2}{4}=\dfrac{(a-b)^2}{4}$; $x+\dfrac{a+b}{2}=\pm\dfrac{a-b}{2}$, etc.

38. $x^2-(b-a)x+\dfrac{(b-a)^2}{4}=\dfrac{(b+a)^2}{4}$; $x-\dfrac{b-a}{2}=\pm\dfrac{b+a}{2}$, etc.

39. $x^2-2ax+a^2=b^2$; $x-a=\pm b$, etc.

40. $x^2-(a^2-a^3)x+\dfrac{(a^2-a^3)^2}{4}=\dfrac{(a^2+a^3)^2}{4}$; $x-\dfrac{a^2-a^3}{2}=\pm\dfrac{a^2+a^3}{2}$, etc.

41. $x^2+\dfrac{ax}{b}+\dfrac{a^2}{4b^2}=\dfrac{9a^2}{4b^2}$; $x+\dfrac{a}{2b}=\pm\dfrac{3a}{2b}$, etc.

42. $x^2-\dfrac{a^2+b^2}{ab}x+\dfrac{(a^2+b^2)^2}{4a^2b^2}=\dfrac{(a^2-b^2)^2}{4a^2b^2}$; $x-\dfrac{a^2+b^2}{2ab}=\pm\dfrac{a^2-b^2}{2ab}$.

XCIII.

1. $\left.\begin{aligned}x^2+2xy+y^2&=1600\\4xy&=1200\end{aligned}\right\}$, $x^2-2xy+y^2=400$, $\left\{\begin{aligned}x+y&=40\\x-y&=\pm 20\end{aligned}\right.$, etc.

2. $\left.\begin{aligned}x^2+2xy+y^2&=169\\4xy&=144\end{aligned}\right\}$, $x^2-2xy+y^2=25$, $\left\{\begin{aligned}x+y&=13\\x-y&=\pm 5\end{aligned}\right.$, etc.

3. $\left.\begin{aligned}x^2+2xy+y^2&=841\\4xy&=400\end{aligned}\right\}$, $x^2-2xy+y^2=441$, $\left\{\begin{aligned}x+y&=29\\x-y&=\pm 21\end{aligned}\right.$, etc.

4. $\left.\begin{aligned}x^2-2xy+y^2&=361\\4xy&=264\end{aligned}\right\}$, $x^2+2xy+y^2=625$, $\left\{\begin{aligned}x-y&=19\\x+y&=\pm 25\end{aligned}\right.$, etc.

5. $\left.\begin{aligned}x^2-2xy+y^2&=2025\\4xy&=1000\end{aligned}\right\}$, $x^2+2xy+y^2=3025$, $\left\{\begin{aligned}x-y&=45\\x+y&=\pm 55\end{aligned}\right.$, etc.

6. $\left.\begin{aligned}x^2-2xy+y^2&=9801\\4xy&=400\end{aligned}\right\}$, $x^2+2xy+y^2=10201$, $\left\{\begin{aligned}x-y&=99\\x+y&=\pm 101\end{aligned}\right.$, etc.

XCIV.

1. $\left.\begin{array}{l} x^2-2xy+y^2=16 \\ x^2 \qquad\quad +y^2=40 \end{array}\right\}$; $2xy=24$; $x^2+2xy+y^2=64$, etc.

2. $\left.\begin{array}{l} x^2-2xy+y^2=100 \\ x^2 \qquad\quad +y^2=178 \end{array}\right\}$; $2xy=78$; $x^2+2xy+y^2=256$, etc.

3. $\left.\begin{array}{l} x^2-2xy+y^2=196 \\ x^2 \qquad\quad +y^2=436 \end{array}\right\}$; $2xy=240$; $x^2+2xy+y^2=676$, etc.

4. $\left.\begin{array}{l} x^2+2xy+y^2=64 \\ x^2 \qquad\quad +y^2=32 \end{array}\right\}$; $2xy=32$; $x^2-2xy+y^2=0$, etc.

5. $\left.\begin{array}{l} x^2+2xy+y^2=144 \\ x^2 \qquad\quad +y^2=104 \end{array}\right\}$; $2xy=40$; $x^2-2xy+y^2=64$, etc.

6. $\left.\begin{array}{l} x^2+2xy+y^2=2401 \\ x^2 \qquad\quad +y^2=1681 \end{array}\right\}$; $2xy=720$; $x^2-2xy+y^2=961$, etc.

XCV.

1. $\left.\begin{array}{l} x^2-\ \ xy+y^2=13 \\ x^2+2xy+y^2=49 \end{array}\right\}$; $3xy=36$; $x^2-2xy+y^2=1$, etc.

2. $\left.\begin{array}{l} x^2-\ \ xy+y^2=\ \ 31 \\ x^2+2xy+y^2=121 \end{array}\right\}$; $3xy=90$; $x^2-2xy+y^2=1$, etc.

3. $\left.\begin{array}{l} x^2-\ \ xy+y^2=\ \ 84 \\ x^2+2xy+y^2=144 \end{array}\right\}$; $3xy=60$; $x^2-2xy+y^2=64$, etc.

4. $\left.\begin{array}{l} x^2+\ \ xy+y^2=28 \\ x^2-2xy+y^2=\ \ 4 \end{array}\right\}$; $3xy=24$; $x^2+2xy+y^2=36$, etc.

5. $\left.\begin{array}{l} x^2+\ \ xy+y^2=49 \\ x^2-2xy+y^2=\ \ 4 \end{array}\right\}$; $3xy=45$; $x^2+2xy+y^2=64$, etc.

6. $\left.\begin{array}{l} x^2+\ \ xy+y^2=93 \\ x^2-2xy+y^2=\ \ 9 \end{array}\right\}$; $3xy=84$; $x^2+2xy+y^2=121$, etc.

XCVI.

1. $\left.\begin{aligned}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}&=\frac{81}{400}\\ \frac{1}{x^2}\qquad+\frac{1}{y^2}&=\frac{41}{400}\end{aligned}\right\}$; $\frac{2}{xy}=\frac{40}{400}$; $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{400}$, etc.

2. $\left.\begin{aligned}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}&=\frac{9}{16}\\ \frac{1}{x^2}\qquad+\frac{1}{y^2}&=\frac{5}{16}\end{aligned}\right\}$; $\frac{2}{xy}=\frac{4}{16}$; $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{16}$, etc.

3. $\left.\begin{aligned}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}&=25\\ \frac{1}{x^2}\qquad+\frac{1}{y^2}&=13\end{aligned}\right\}$; $\frac{2}{xy}=12$; $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=1$, etc.

4. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{12}$, etc.

5. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{2}$, etc.

6. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=7$, etc.

XCVII.

1. $x-y=1$; $x^2-2xy+y^2=1$; $3xy=36$, etc.

2. $x^2+12xy+36y^2=576$; $x+6y=\pm 24$.

   Now, from the first equation, $x(x+6y)=144$, $\therefore \pm 24x=144$, etc.

3. $x^2+2xy+y^2=441$; $x+y=\pm 21$.

   Now, from the first equation, $x(x+y)=210$, $\therefore \pm 21x=210$, etc.

4. $x^2+2xy+y^2=100$; and $x^2-2xy+y^2=36$;

   hence $x+y=\pm 10$, and $x-y=\pm 6$, etc.

5. $x+y=8$; $x^2+2xy+y^2=64$; $3xy=45$, etc.

6. $4x=10+5y$; put $\dfrac{10+5y}{4}$ for x in the first equation;

then $\dfrac{(10+5y)^2}{4}+\dfrac{90y+45y^2}{4}=100$; $70y^2+190y=660$;

$7y^2+19y=66$; whence $y=2$ or $-\dfrac{33}{7}$, etc.

7. Put $y=mx$. Then $\dfrac{x^2+mx^2+m^2x^2}{3m^2x^2-5mx^2}=\dfrac{39}{25}$; $\dfrac{1+m+m^2}{3m^2-5m}=\dfrac{39}{25}$;

$25+25m+25m^2=117m^2-195m$; $92m^2-220m=25$;

hence we get $m=\dfrac{5}{2}$ or $-\dfrac{5}{46}$. Taking the value $\dfrac{5}{2}$, we get

$x^2+\dfrac{5x^2}{2}+\dfrac{25x^2}{4}=39$; $39x^2=39\times 4$; $x=\pm 2$, etc.

8. Put $y=mx$. Then $\dfrac{x^2+mx^2}{mx^2-m^2x^2}=\dfrac{66}{5}$; $\dfrac{1+m}{m-m^2}=\dfrac{66}{5}$;

$5+5m=66m-66m^2$; $m^2-\dfrac{61m}{66}=-\dfrac{5}{66}$; hence we get

$m=\dfrac{1}{11}$, or $\dfrac{5}{6}$. Taking the value $\dfrac{5}{6}$, $x^2+\dfrac{5x^2}{6}=66$,

whence $x^2=36$, $x=\pm 6$, etc.

9. Put $y=mx$. Then $\dfrac{3+4m}{5m+2m^2}=\dfrac{20}{12}$; whence $m=\dfrac{1}{2}$ or $-\dfrac{3}{10}$.

Taking the value $\frac{1}{2}$, we get $3x^2+2x^2=20$; $x=\pm 2$, etc.

10. Put $y=mx$. Then $\dfrac{x^2-mx^2+m^2x^2}{3x^2+13mx^2+8m^2x^2}=\dfrac{7}{162}$; $\dfrac{1-m+m^2}{3+13m+8m^2}=\dfrac{7}{162}$

$162-162m+162m^2=21+91m+56m^2$; $106m^2-253m=-141$;

$m^2-\dfrac{253}{136}m=-\dfrac{141}{106}$; $m-\dfrac{253}{212}=\pm\dfrac{65}{212}$; $m=\dfrac{3}{2}$ or $-\dfrac{47}{53}$.

Taking the value $\dfrac{3}{2}$, we get $x^2-\dfrac{3x^2}{2}+\dfrac{9x^2}{4}=7$; $x=\pm 2$, etc.

11. Put $y=mx$. Then $\frac{1-m}{m+m^2}=\frac{35}{18}$; $35m^2+53m=18$;

$m+\frac{53}{70}=\pm\frac{73}{70}$; $m=\frac{2}{7}$ or $-\frac{9}{10}$.

Taking the value $\frac{2}{7}$ we get $x=\pm 7$, etc.

12. $x=\frac{29-7y}{5}$; putting this for x in the first equation, we get

$\frac{2523-1218y+147y^2}{25}+\frac{116y-28y^2}{5}+5y^2=71$;

$132y^2-638y=-748$; $y=2$ or $\frac{17}{6}$, etc.

13. $x+y=22$; $x^2+2xy+y^2=484$; $3xy=360$; $xy=120$;

$x^2-2xy+y^2=4$; $x-y=\pm 2$, etc.

14. Subtract the second equation from the first; $x^2+2xy+y^2=169$;

$x+y=\pm 13$. Take the positive value, and for y put $13-x$ in the first equation; $x^2+117x-9x^2=340$; $x^2-\frac{117}{8}x=-\frac{340}{8}$;

$x^2-\frac{117}{8}x+\frac{13689}{256}=\frac{2809}{256}$; $x-\frac{117}{16}=\pm\frac{53}{16}$, etc.

15. $\left.\begin{array}{r} x^2+y^2=225 \\ 2xy=216 \end{array}\right\}$, $\left.\begin{array}{r} x^2+y^2+2xy=441 \\ x^2+y^2-2xy=9 \end{array}\right\}$, $\left.\begin{array}{r} x+y=\pm 21 \\ x-y=\pm 3 \end{array}\right\}$, etc.

XCVIII.

1. Let x be the number; then $\frac{x}{2}\times\frac{x}{3}=864$; $x^2=5184$, etc.

2. Let x be the number; then $\left(\frac{x}{7}\times\frac{x}{8}\right)\div 3=298\frac{2}{3}$; $\frac{x^2}{56}=896$, etc.

3. Let x be the number; then $(94-x)(94+x)=8512$, etc.

4. Let x be the greater; then $\frac{750}{x}$ is the less; and $x \times \frac{x}{750} = 3\frac{1}{3}$, etc.

5. Let x and y be the numbers; then $x^2 + y^2 = 13001$ and $x^2 - y^2 = 1449$; adding, $2x^2 = 14450$, etc.

6. Let x be the greater; then $\frac{377}{x}$ is the less; and $x - 21 = 21 - \frac{377}{x}$;

$x^2 - 42x = -377$; $x^2 - 42x + 441 = 64$, etc.

7. Let x be the number; then $\frac{x}{2} \times \frac{x}{3} \times \frac{x}{4} \times \frac{x}{5} = 6750$; $x^4 = 810000$;

$x^2 = 900$; $x = 30$.

8. Let x be the number; then $\frac{11500}{x} = x + \frac{51}{x}$; $11500 = x^2 + 51$, etc.

9. Let x be the number; then $(x+20)^2 + 2(x-10)^2 = 17475$;

$x^2 + 40x + 400 + 2x^2 - 40x + 200 = 17475$; $3x^2 = 16875$, etc.

10. Let x and $26 - x$ be the numbers; then $x^2 + (26-x)^2 = 436$;

$2x^2 - 52x + 676 = 436$; $x^2 - 26x = -120$, etc.

11. Let $x + 17$ and x be the numbers; then $(x+17)^2 + x^2 = 325$, etc.

12. Let x and $\frac{255}{x}$ be the numbers; then $x^2 + \frac{65025}{x^2} = 514$;

$x^4 - 514x^2 = -65025$; $x^4 - 514x^2 + 66049 = 1024$;

$x^2 - 257 = \pm 32$; $x^2 = 289$ or 225; $x = 17$ or 15.

13. Let x and $16 - x$ be the parts; then $x^2 + (16-x)^2 + (16-x)x = 208$

$x^2 - 16x + 256 = 208$; $x^2 - 16x = -48$, etc.

14. Let x^2 be the number; then $x^2 + x = 1332$; $x^2 + x + \frac{1}{4} = \frac{5329}{4}$

$x + \frac{1}{2} = \pm\frac{73}{2}$; $x = 36$; $x^2 = 1296$.

15. Let x^2 be the number; then $x^2-x=\frac{195}{4}$, etc.

16. Let x^2 be the number; then $x^2-x=2550$, etc.

17. Let x and $\frac{24}{x}$ be the numbers; then $\left(x+\frac{24}{x}\right)\left(x-\frac{24}{x}\right)=20$;

$x^2-\frac{576}{x^2}=20$; $x^4-20x^2=576$; $x^4-20x^2+100=676$;

$x^2-10=\pm 26$; taking the positive value $x=6$, etc.

18. Let x be the greater, y the less; then $x(x+y)=204$ and

$y(x-y)=35$, or, $x^2+xy=204$ and $xy-y^2=35$.

Put $y=mx$. Then $\frac{x^2+mx^2}{mx^2-m^2x^2}=\frac{204}{35}$; hence one value of m is $\frac{5}{12}$

and $x^2+\frac{5x^2}{12}=204$; $17x^2=2448$; $x^2=144$, etc.

19. Let x and $x-5$ be the numbers; then $x(2x-5)=228$, etc.

20. Let $x-1$, x, $x+1$ be the numbers; then $(x-1)x(x+1)=3x$;

$x^2-1=3$; $x^2=4$; $x=2$, etc.

21. Let x and $x+1$ be the numbers; then $(x+1)^2-x^2=15$, etc.

22. Let x and $x+1$ be the numbers; then $x^2+(x+1)^2=481$, etc.

23. Let $x-1$, x and $x+1$ be the numbers; then $(x-1)^2+x^2+(x+1)^2$

$=365$, etc.

24. Let x be the number; then $\frac{60}{x}=\frac{60}{x+3}+1$;

$60x+180=60x+x^2+3x$; $x^2+3x=180$, etc.

25. Let x be the number; then $\frac{80}{x}=\frac{80}{x+4}+1$, etc.

26. Let x be the price of each piece in shillings; then $675 = 48 \times \frac{675}{x} - x$;
$675x = 32400 - x^2$, etc.

27. Let x be the number; then $\frac{180}{x} = \frac{180}{x+3} + 3$, etc.

28. Let x be the number of miles; then $\frac{108}{x} = \frac{108}{x+3} + 6$, etc.

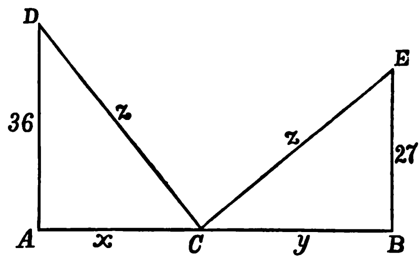
29. Let x be the number of sheep; then $\frac{60}{x} = \frac{54}{x-15} - \frac{1}{10}$, etc.

30. Let x and $x+2$ be the number of hours; then $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{2\frac{11}{12}}$, etc.

31. Let x be the number of yards in the breadth;
then $x(x+1) = 10100$, etc.

32. Let $10x + y$ be the number; then $x = 2y$ and $(10x+y)(10y+x)$
$= 2268$; hence $21y \times 12y = 2268$; $y^2 = 9$, etc.

33. Let x and y be the distances in feet of the ladder from the bottoms of the houses; z the length of the ladder in feet.



Then since DCE is a right angle,

DCA and ECB together make a right angle. Eucl. i. 13.

But DCA and ADC together make a right angle. Eucl. i. 32.

$\therefore$ angle $ECB =$ angle ADC.

Hence triangles ACD and BEC are equal in all respects. Eucl. i. 26.

$\therefore x = 27$, and $y = 36$; and $x + y = 63$.

Also $z^2 = (27)^2 + (36)^2$. Eucl. i. 47; and $\therefore z = 45$.

34. [illegible]

Then [illegible]

and [illegible]

[illegible]

35. Let $10x + y$ be the number. Since the number is [illegible], and the difference of the digits [illegible] the product of the digits.

Then $y - x =$ [illegible]

[illegible]

36. Let x be the number of rows. [illegible] is then the number of trees in each row.

Then [illegible]

or [illegible]

or $x^2 -$ [illegible], etc.

37. Let x be the number of men in the regiment, y the number in the side of the square at the first attempt.

Then $y^2 = x -$ [illegible] and [illegible]

Hence [illegible], whence [illegible], etc.

[illegible]

[illegible]

Then [illegible]

and [illegible]

[illegible]

2. $x=\dfrac{92-19y}{7}=13-2y-\dfrac{5y-1}{7}$; let $\dfrac{5y-1}{7}=m$

$y=\dfrac{7m+1}{5}=m+\dfrac{2m+1}{5}$; let $\dfrac{2m+1}{5}=n$

$m=\dfrac{5n-1}{2}=2n+\dfrac{n-1}{2}$; let $\dfrac{n-1}{2}=p$

$n=2p+1$

Then $y=m+n=2n+p+n=4p+2+p+2p+1=7p+3$

$x=13-14p-6-4p-2-p=5-19p$

when $p=0$, $x=5$ and $y=3$.

3. $x=\dfrac{1170-19y}{13}=90-y-\dfrac{6y}{13}$; let $\dfrac{6y}{13}=m$

$y=\dfrac{13m}{6}=2m+\dfrac{m}{6}$; let $\dfrac{m}{6}=n$; $m=6n$

Then $y=13n$ and $x=90-19n$

when $n=0$, $y=0$ and $x=90$

$n=1$, $y=13$ and $x=71$, etc.

4. $x=\dfrac{26-5y}{3}=8-y-\dfrac{2y-2}{3}$; let $\dfrac{2y-2}{3}=m$

$y=\dfrac{3m+2}{2}=m+1+\dfrac{m}{2}$; let $\dfrac{m}{2}=n$; $m=2n$

Then $y=2n+1+n=3n+1$

$x=8-3n-1-2n=7-5n$

when $n=0$, $y=1$, $x=7$

$n=1$, $y=4$, $x=2$.

5. $5y=14x-7$; $y=2x-1+\dfrac{4x-2}{5}$; let $\dfrac{4x-2}{5}=m$

$x=\dfrac{5m+2}{4}=m+\dfrac{m+2}{4}$; let $\dfrac{m+2}{4}=n$; $m=4n-2$

Then $x=4n-2+n=5n-2$

$y=10n-4-1+4n-2=14n-7$, etc.

6. $x=\frac{1031-15y}{11}=93-y-\frac{4y-8}{11}$; let $\frac{4y-8}{11}=m$

$y=\frac{11m+8}{4}=2m+2+\frac{3m}{4}$; let $\frac{3m}{4}=n$

$m=\frac{4n}{3}=n+\frac{n}{3}$; let $\frac{n}{3}=p$; $n=3p$

Then $y=8p+2+3p=11p+2$

$x=93-11p-2-4p=91-15p$, etc.

7. $y=\frac{308-11x}{7}=44-x-\frac{4x}{7}$; let $\frac{4x}{7}=m$

$x=\frac{7m}{4}=m+\frac{3m}{4}$; let $\frac{3m}{4}=n$

$m=\frac{4n}{3}=n+\frac{n}{3}$; let $\frac{n}{3}=p$; $n=3p$

Then $x=m+n=3p+p+3p=7p$

$y=44-x-m=44-7p-4p=44-11p$, etc.

8. $x=\frac{23+19y}{4}=5+4y+\frac{3y+3}{4}$; let $\frac{3y+3}{4}=m$

$y=\frac{4m-3}{3}=m-1+\frac{m}{3}$; let $\frac{m}{3}=n$; $m=3n$

Then $y=3n-1+n=4n-1$

$x=5+16n-4+3n=19n+1$, etc.

9. $y=\frac{20x-683}{9}=2x-75+\frac{2x-8}{9}$; let $\frac{2x-8}{9}=m$

$x=\frac{9m+8}{2}=4m+4+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $x=8n+4+n=9n+4$

$y=18n+8-75+2n=20n-67$, etc.

10. $x=\frac{383-7y}{3}=127-2y-\frac{y-2}{3}$; let $\frac{y-2}{3}=m$

Then $y=3m+2$

$x=127-6m-4-m=123-7m$. etc.

11. $y=\dfrac{54-27x}{4}=13-6x-\dfrac{3x-2}{4}$; let $\dfrac{3x-2}{4}=m$

$x=\dfrac{4m+2}{3}=m+\dfrac{m+2}{3}$; let $\dfrac{m+2}{3}=n$; $m=3n-2$

Then $x=3n-2+n=4n-2$

$y=13-24n+12-3n+2=27-27n$, etc.

12. $x=\dfrac{653-9y}{7}=93-y-\dfrac{2y-2}{7}$; let $\dfrac{2y-2}{7}=m$

$y=\dfrac{7m+2}{2}=3m+1+\dfrac{m}{2}$; let $\dfrac{m}{2}=n$; $m=2n$

Then $y=6n+1+n=7n+1$

$x=93-7n-1-2n=92-9n$, etc.

13. $\dfrac{x}{7}+\dfrac{y}{9}=\dfrac{57}{63}$; $9x+7y=57$

$y=\dfrac{57-9x}{7}=8-x-\dfrac{2x-1}{7}$; let $\dfrac{2x-1}{7}=m$

$x=\dfrac{7m+1}{2}=3m+\dfrac{m+1}{2}$; let $\dfrac{m+1}{2}=n$; $m=2n-1$

Then $x=6n-3+n=7n-3$

$y=8-7n+3-2n+1=12-9n$

when $n=1$, $x=4$, $y=3$.

14. $\dfrac{x}{11}-\dfrac{y}{13}=\dfrac{82}{143}$; $13x-11y=82$

$y=\dfrac{13x-82}{11}=x-7+\dfrac{2x-5}{11}$; let $\dfrac{2x-5}{11}=m$

$x=\dfrac{11m+5}{2}=5m+2+\dfrac{m+1}{2}$; let $\dfrac{m+1}{2}=n$; $m=2n-1$

Then $x=10n-5+2+n=11n-3$

$y=11n-3-7+2n-1=13n-11$

when $n=1$, $x=8$, $y=2$.

15. Let x be the number of florins, y the number of half-crowns; then

$4x+5y=58$; $x=14-y-\frac{y-2}{4}$; let $\frac{y-2}{4}=m$

Then $y=4m+2$ and $x=12-5m$, etc.

16. Let x be the number of half-guineas, y the number of half-crowns; then

$21x+5y=800$; $y=160-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$ and $y=160-21m$

when $m=1$, $x=5$, $y=139$, etc.

17. Let x be the number; y the first quotient, z the second quotient.

Then $\frac{x}{5}=y+\frac{2}{5}$; $\frac{x}{9}=z+\frac{3}{9}$

Hence $5y+2=9z+3$; $y=z+\frac{4z+1}{5}$; let $\frac{4z+1}{5}=m$

$z=\frac{5m-1}{4}=m+\frac{m-1}{4}$; let $\frac{m-1}{4}=n$; $m=4n+1$

$z=5n+1$; $y=9n+2$

Hence $y=2, 11 \ldots$ and hence $x=12, 57 \ldots$

18. Let x be the number of guineas, and y the number of crowns.

Then $21x+5y=235$; $y=47-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$; and $y=47-21m$, etc.

19. Let x be the number of half-guineas, and y the number of half-crowns.

Then $21x+5y=183$; $y=36-4x-\frac{x-3}{5}$ let $\frac{x-3}{5}=m$

Then $x=5m+3$; and $y=36-20m-12-m=24-21m$, etc.

20. Divide by 17; then $19x-31y=\frac{1000}{17}$; and the right-hand side being a fraction, no integral values of x and y can make the left side equal to the right.

21. Let x be the number of oxen, y of sheep, z of hens.

Then $x+y+z=100$, and $100x+20y+z=2000$,

$\therefore 99x+19y=1900$

Hence $y=100-5x-\frac{4x}{19}$; let $\frac{4x}{19}=m$

$x=\frac{19m}{4}=5m-\frac{m}{4}$; let $\frac{m}{4}=n$; $m=4n$

Then $x=19n$; $y=100-99n$

If $n=1$, $x=19$, $y=1$, and $\therefore z=80$.

22. Let A give x sixpences and receive y fourpenny pieces.

Then $6x-4y=58$; or $3x-2y=29$

Then $y=x-14+\frac{x-1}{2}$; let $\frac{x-1}{2}=m$

Then $x=2m+1$ and $y=3m-13$, etc.

23. Let x be the number of half-crowns, y of florins, z of shillings.

Then $x+y=4z$, and $\frac{5x}{2}+2y+z=244$

Hence $x+y=4z$, and $10x+8y+4z=976$

$\therefore 11x+9y=976$

Hence $y=108-x-\frac{2x-4}{9}$; let $\frac{2x-4}{9}=m$

$x=4m+2+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $x=9n+2$; $y=108-9n-2-2n=106-11n$

Hence $x=2$, $y=106$, $z=27$, which gives the greatest number of coins possible.

24. Let there be x half-crowns, y florins, and z fourpenny-pieces.

Then $x+y+z=50$, and $30x+24y+4z=1200$.

Hence $26x+20y=1000$.

$\therefore y=50-x-\dfrac{3x}{10}$; let $\dfrac{3x}{10}=m$.

Then $x=\dfrac{10m}{3}=3m+\dfrac{m}{3}$; let $\dfrac{m}{3}=n$; $m=3n$.

Hence $x=10n$ and $y=50-13n$

$\therefore x=10, 20, 30$; $y=37, 24, 11$; $z=3, 6, 9$.

25. Let A give x sovereigns, and receive y dollars.

Then $240x-51y=12$; or, $80x-17y=4$.

$$\text{Hence } y=4x+\frac{12x-4}{17}; \text{ let } \frac{12x-4}{17}=m$$

$$x=m+\frac{5m+4}{12}; \text{ let } \frac{5m+4}{12}=n$$

$$m=2n+\frac{2n-4}{5}; \text{ let } \frac{2n-4}{5}=p$$

$$n=2p+2+\frac{p}{2}; \text{ let } \frac{p}{2}=q; p=2q.$$

Then $x=m+n=3n+p=6p+6+5q=17q+6$

$y=4x+m=68q+24+2n+p=82q+28$.

Hence when $q=0$, $x=6$, $y=28$.

26. Let $2x$ and $3y$ be the parts; then $2x+3y=25$.

$$\text{Then } x=12-y-\frac{y-1}{2}; \text{ let } \frac{y-1}{2}=m$$

$$y=2m+1 \text{ and } x=11-3m$$

When $m=0$, $y=1$, $x=11$; $2x=22$, $3y=3$, etc.

27. Let there be x crowns and y florins; then $5x+2y=49$.

$$\text{Then } y=24-2x-\frac{x-1}{2}; \text{ let } \frac{x-1}{2}=m$$

$$x=2m+1, \text{ and } y=22-5m$$

Hence we get positive values for x and y, when $m=0, 1, 2, 3, 4$.

28. Let $7x$ and $11y$ be the parts; then $7x+11y=100$.

Then $x=14-y-\frac{4y-2}{7}$; let $\frac{4y-2}{7}=m$.

$$y=\frac{7m+2}{4}=m+\frac{3m+2}{4}\text{; let }\frac{3m+2}{4}=n$$

$$m=\frac{4n-2}{3}=n+\frac{n-2}{3}\text{; let }\frac{n-2}{3}=p\text{; }n=3p+2.$$

Hence $y=2n+p=7p+4$

$x=14-7p-4-n-p=8-11p$.

When $p=0$, $y=4$, $x=8$; $7x=56$, $11y=44$.

29. Let the quotients be x and y.

Then $5x+2$ and $7y+4$ are the dividends

$\therefore 5x+7y+6=100$, or, $5x+7y=94$.

Then $x=18-y-\frac{2y-4}{5}$; let $\frac{2y-4}{5}=m$

$$y=2m+2+\frac{m}{2}\text{; let }\frac{m}{2}=n\text{; }m=2n.$$

Then $y=5n+2$ and $x=16-7n$.

Hence $x=16, 9, 2$; and $y=2, 7, 12$

and $5x+2=82, 47, 12$; and $7y+4=18, 53, 88$.

30. The L. C. M. of 2, 3, 4, 5, 6 is 60.

Hence the general expression for the number required is $60m+1$.

Making $m=1, 2, 3 \dots$ successively, we get 61, 121, 181, 241, 301, 361, and of these 301 is the only multiple of 7.

CI.

1.

$$\begin{array}{l}
x^{2p}+x^py^p+y^{2p} \\
x^{2p}-x^py^p+y^{2p} \\
\hline
x^{4p}+x^{3p}y^p+x^{2p}y^{2p} \\
\qquad -x^{3p}y^p-x^{2p}y^{2p}-x^py^{3p} \\
\qquad\qquad +x^{2p}y^{2p}+x^py^{3p}+y^{4p} \\
\hline
x^{4p}\qquad\quad +x^{2p}y^{2p}\qquad\quad +y^{4p}
\end{array}$$

2.
$$\begin{array}{l}
a^{3m}+3a^{2m}y^n+9a^my^{2n}+27y^{3n}\\
\underline{a^m-3y^n\qquad\qquad\qquad\qquad\qquad}\\
a^{4m}+3a^{3m}y^n+9a^{2m}y^{2n}+27a^my^{3n}\\
\qquad -3a^{3m}y^n-9a^{2m}y^{2n}-27a^my^{3n}-81y^{4n}\\
\hline
a^{4m}\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -81y^{4n}
\end{array}$$

3.
$$\begin{array}{l}
x^{4d}-2ax^{2d}+4a^2\\
\underline{x^{4d}+2ax^{2d}+4a^2}\\
x^{8d}-2ax^{6d}+4a^2x^{4d}\\
\qquad +2ax^{6d}-4a^2x^{4d}+8a^3x^{2d}\\
\qquad\qquad\qquad +4a^2x^{4d}-8a^3x^{2d}+16a^4\\
\hline
x^{8d}\qquad\qquad +4a^2x^{4d}\qquad\qquad +16a^4
\end{array}$$

4.
$$\begin{array}{l}
a^m+b^n+c^r\\
\underline{a^m-b^n+c^r}\\
a^{2m}+a^mb^n+a^mc^r\\
\qquad -a^mb^n-b^{2n}-b^nc^r\\
\qquad\qquad\qquad +a^mc^r+b^nc^r+c^{2r}\\
\hline
a^{2m}-b^{2n}+c^{2r}+2a^mc^r
\end{array}$$

5.
$$\begin{array}{l}
a^m+b^n-2c^r\\
\underline{2a^m-b+c^2}\\
2a^{2m}+2a^mb^n-4a^mc^r\\
\qquad -a^mb-b^{n+1}+2bc^r\\
\qquad\qquad +a^mc^2+b^nc^2-2c^{r+2}\\
\hline
2a^{2m}+2a^mb^n-4a^mc^r-a^mb-b^{n+1}+2bc^r+a^mc^2+b^nc^2-2c^{r+2}
\end{array}$$

6.
$$\begin{array}{l}
x^{mn-n}-y^m\\
\underline{x^n+y^{mn-n}}\\
x^{mn}-x^ny^m\\
\qquad +x^{mn-n}.y^{mn-n}-y^{mn-n+m}\\
\hline
x^{mn}+x^{mn-n}.y^{mn-n}-x^ny^m-y^{mn-n+m}
\end{array}$$

7.
$$\begin{array}{l}
x^{2n} - x^n y^n + y^{2n} \\
\underline{x^{2n} + x^n y^n + y^{2n}} \\
x^{4n} - x^{3n}y^n + x^{2n}y^{2n} \\
\qquad + x^{3n}y^n - x^{2n}y^{2n} + x^n y^{3n} \\
\qquad\qquad + x^{2n}y^{2n} - x^n y^{3n} + y^{4n} \\
\hline
x^{4n} \qquad + x^{2n}y^{2n} \qquad + y^{4n}
\end{array}$$

8.
$$\begin{array}{l}
a^{p^2+p} - b^{p^2} + c^p \\
\underline{a^{p^2-p} + b^{1-p^2} + c^{1-p}} \\
a^{2p^2} - a^{p^2-p}b^{p^2} + a^{p^2-p}c^p \\
\qquad + a^{p^2+p}b^{1-p^2} - b + b^{1-p^2}c^p \\
\qquad\qquad + a^{p^2+p}c^{1-p} - b^{p^2}c^{1-p} + c \\
\hline
a^{2p^2} - a^{p^2-p}b^{p^2} + a^{p^2-p}c^p + a^{p^2+p}b^{1-p^2} - b + b^{1-p^2}c^p \\
\qquad\qquad + a^{p^2+p}c^{1-p} - b^{p^2}c^{1-p} + c
\end{array}$$

9.
$$\begin{array}{l}
x^{2p} + x^p + 1 \\
\underline{x^{2p} + x^p + 1} \\
x^{4p} + x^{3p} + x^{2p} \\
\qquad + x^{3p} + x^{2p} + x^p \\
\qquad\qquad + x^{2p} + x^p + 1 \\
\hline
x^{4p} + 2x^{3p} + 3x^{2p} + 2x^p + 1
\end{array}$$

10.
$$\begin{array}{l}
x^{2p} - x^p + 1 \\
\underline{x^{2p} - x^p + 1} \\
x^{4p} - x^{3p} + x^{2p} \\
\qquad - x^{3p} + x^{2p} - x^p \\
\qquad\qquad + x^{2p} - x^p + 1 \\
\hline
x^{4p} - 2x^{3p} + 3x^{2p} - 2x^p + 1
\end{array}$$

CII.

1.
$$\begin{array}{l}
x^m - y^m)\,x^{4m} - y^{4m}\,(x^{3m} + x^{2m}y^m + x^m y^{2m} + y^{3m} \\
\qquad\quad \underline{x^{4m} - x^{3m}y^m} \\
\qquad\qquad\quad x^{3m}y^m - y^{4m} \\
\qquad\qquad\quad \underline{x^{3m}y^m - x^{2m}y^{2m}} \\
\qquad\qquad\qquad\qquad x^{2m}y^{2m} - y^{4m} \\
\qquad\qquad\qquad\qquad \underline{x^{2m}y^{2m} - x^m y^{3m}} \\
\qquad\qquad\qquad\qquad\qquad x^m y^{3m} - y^{4m} \\
\qquad\qquad\qquad\qquad\qquad \underline{x^m y^{3m} - y^{4m}}
\end{array}$$

2. $x^n + y^n)\,x^{5n} + y^{5n}\,(x^{4n} - x^{3n}y^n + x^{2n}y^{2n} - x^ny^{3n} + y^{4n}$

$$\begin{array}{l}
x^{5n} + x^{4n}y^n \\
\hline
-x^{4n}y^n + y^{5n} \\
-x^{4n}y^n - x^{3n}y^{2n} \\
\hline
x^{3n}y^{2n} + y^{5n} \\
x^{3n}y^{2n} + x^{2n}y^{3n} \\
\hline
-x^{2n}y^{3n} + y^{5n} \\
-x^{2n}y^{3n} - x^ny^{4n} \\
\hline
x^ny^{4n} + y^{5n} \\
x^ny^{4n} + y^{5n} \\
\hline
\end{array}$$

3. $x^r - y^r)\,x^{6r} - y^{6r}\,(x^{5r} + x^{4r}y^r$, etc.

$$\begin{array}{l}
x^{6r} - x^{5r}y^r \\
\hline
x^{5r}y^r - y^{6r}, \text{ etc.}
\end{array}$$

4. $a^{3p} + b^{2q})\,a^{15p} + b^{10q}\,(a^{12p} - a^{9p}b^{2q}$, etc.

$$\begin{array}{l}
a^{15p} + a^{12p}b^{2q} \\
\hline
-a^{12p}b^{2q} + b^{10q}, \text{ etc.}
\end{array}$$

5. $x^d - 3)\,x^{5d} - 243)\,x^{4d} + 3x^{3d}$, etc.

$$\begin{array}{l}
x^{5d} - 3x^{4d} \\
\hline
3x^{4d} - 243, \text{ etc.}
\end{array}$$

6. $a^{2m} + 2a^mx^n + 4x^{2n})\,a^{4m} + 4a^{2m}x^{2n} + 16x^{4n}\,(a^{2m} - 2a^mx^n + 4x^{2n}$

$$\begin{array}{l}
a^{4m} + 2a^{3m}x^n + 4a^{2m}x^{2n} \\
\hline
-2a^{3m}x^n + 16x^{4n} \\
-2a^{3m}x^n - 4a^{2m}x^{2n} - 8a^mx^{3n} \\
\hline
4a^{2m}x^{2n} + 8a^mx^{3n} + 16x^{4n} \\
4a^{2m}x^{2n} + 8a^mx^{3n} + 16x^{4n} \\
\hline
\end{array}$$

7. $1+5x^p+x^{2p})\,2+9x^p+14x^{3p}+3x^{4p}\,(2-x^p+3x^{2p}$

$$2+10x^p+2x^{2p}$$

$$-x^p-2x^{2p}+14x^{3p}$$

$$-x^p-5x^{2p}-x^{3p}$$

$$3x^{2p}+15x^{3p}+3x^{4p}$$

$$3x^{2p}+15x^{3p}+3x^{4p}$$

8. $b^mc^{2m}-2b^{2m}c^m+b^{3m})\,4b^{2m}c^{3m}-13b^{3m}c^{2m}+14b^{4m}c^m-5b^{5m}\,(4b^mc^m-5b^{2m}$

$$4b^{2m}c^{3m}-8b^{3m}c^{2m}+4b^{4m}c^m$$

$$-5b^{3m}c^{2m}+10b^{4m}c^m-5b^{5m}$$

$$-5b^{3m}c^{2m}+10b^{4m}c^m-5b^{5m}$$

9. $a^{6m}+6a^{5m}+15a^{4m}+20a^{3m}+15a^{2m}+6a^m+1\,(a^{3m}+3a^{2m}+3a^m+1$

$$a^{6m}$$

$2a^{3m}+3a^{2m}$

$$6a^{5m}+15a^{4m}$$

$$6a^{5m}+9a^{4m}$$

$2a^{3m}+6a^{2m}+3a^m$

$$6a^{4m}+20a^{3m}+15a^{2m}$$

$$6a^{4m}+18a^{3m}+9a^{2m}$$

$2a^{3m}+6a^{2m}+6a^m+1$

$$2a^{3m}+6a^{2m}+6a^m+1$$

$$2a^{3m}+6a^{2m}+6a^m+1$$

10. $a^{2m}+2a^mb^n+b^{2n}+2a^mc^r+2b^nc^r+c^{2r}\,(a^m+b^n+c^r$

$$a^{2m}$$

$2a^m+b^n$

$$2a^mb^n+b^{2n}$$

$$2a^mb^n+b^{2n}$$

$2a^m+2b^n+c^r$

$$2a^mc^r+2b^nc^r+c^{2r}$$

$$2a^mc^r+2b^nc^r+c^{2r}$$

CIII.

1.
$$\begin{array}{l}
x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1 \\
x^{\frac{1}{3}} - 1 \\
\hline
x - 2x^{\frac{2}{3}} + x^{\frac{1}{3}} \\
\quad - x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 1 \\
\hline
x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1
\end{array}$$

2.
$$\begin{array}{l}
y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1 \\
y^{\frac{1}{4}} - 1 \\
\hline
y + y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} \\
\quad - y^{\frac{3}{4}} - y^{\frac{1}{2}} - y^{\frac{1}{4}} - 1 \\
\hline
y \qquad\qquad\qquad\quad - 1
\end{array}$$

3.
$$\begin{array}{l}
a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}} \\
a^{\frac{2}{3}} - x^{\frac{2}{3}} \\
\hline
a^2 + a^{\frac{4}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{4}{3}} \\
\quad - a^{\frac{4}{3}}x^{\frac{2}{3}} - a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2 \\
\hline
a^2 \qquad\qquad\qquad\quad - x^2
\end{array}$$

4.
$$\begin{array}{l}
a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}} \\
a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \\
\hline
a - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}c^{\frac{2}{3}} \\
\quad + a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{2}{3}}c^{\frac{1}{3}} + b^{\frac{1}{3}}c^{\frac{2}{3}} \\
\qquad\qquad + a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{2}{3}} + c \\
\hline
a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}
\end{array}$$

5.
$$\begin{array}{l}
5x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{1}{3}} + 7y^{\frac{2}{3}} \\
2x^{\frac{1}{3}} - 3y^{\frac{1}{3}} \\
\hline
10x + 4x^{\frac{2}{3}}y^{\frac{1}{3}} + 6x^{\frac{1}{3}}y^{\frac{1}{3}} + 14x^{\frac{1}{3}}y^{\frac{2}{3}} \\
\qquad - 15x^{\frac{2}{3}}y^{\frac{1}{3}} - 6x^{\frac{1}{3}}y^{\frac{2}{3}} - 9x^{\frac{1}{3}}y^{\frac{2}{3}} - 21y \\
\hline
10x - 11x^{\frac{2}{3}}y^{\frac{1}{3}} \qquad\qquad + 5x^{\frac{1}{3}}y^{\frac{2}{3}} - 21y
\end{array}$$

6.
$$\begin{array}{l}
m^{\frac{4}{5}} + m^{\frac{3}{5}}n^{\frac{1}{5}} + m^{\frac{2}{5}}n^{\frac{2}{5}} + m^{\frac{1}{5}}n^{\frac{3}{5}} + n^{\frac{4}{5}} \\
m^{\frac{1}{5}} - n^{\frac{1}{5}} \\
\hline
m + m^{\frac{4}{5}}n^{\frac{1}{5}} + m^{\frac{3}{5}}n^{\frac{2}{5}} + m^{\frac{2}{5}}n^{\frac{3}{5}} + m^{\frac{1}{5}}n^{\frac{4}{5}} \\
\quad - m^{\frac{4}{5}}n^{\frac{1}{5}} - m^{\frac{3}{5}}n^{\frac{2}{5}} - m^{\frac{2}{5}}n^{\frac{3}{5}} - m^{\frac{1}{5}}n^{\frac{4}{5}} - n \\
\hline
m \qquad\qquad\qquad\qquad\qquad\qquad - n
\end{array}$$

7.

$$\begin{array}{l}
m^{\frac{2}{3}} - 2d^{\frac{1}{4}}m^{\frac{1}{3}} + 4d^{\frac{1}{2}} \\
m^{\frac{2}{3}} + 2d^{\frac{1}{4}}m^{\frac{1}{3}} + 4d^{\frac{1}{2}} \\
\hline
m^{\frac{4}{3}} - 2d^{\frac{1}{4}}m + 4d^{\frac{1}{2}}m^{\frac{2}{3}} \\
\quad + 2d^{\frac{1}{4}}m - 4d^{\frac{1}{2}}m^{\frac{2}{3}} + 8d^{\frac{3}{4}}m^{\frac{1}{3}} \\
\qquad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} - 8d^{\frac{3}{4}}m^{\frac{1}{3}} + 16d \\
\hline
m^{\frac{4}{3}} + 4d^{\frac{1}{2}}m^{\frac{2}{3}} + 16d
\end{array}$$

8.

$$\begin{array}{l}
8a^{\frac{3}{4}} + 4a^{\frac{1}{2}}b^{\frac{1}{4}} + 5a^{\frac{1}{4}}b^{\frac{1}{2}} + 9b^{\frac{3}{4}} \\
2a^{\frac{1}{4}} - 3b^{\frac{1}{4}} \\
\hline
16a + 8a^{\frac{3}{4}}b^{\frac{1}{4}} + 10a^{\frac{1}{2}}b^{\frac{1}{2}} + 18a^{\frac{1}{4}}b^{\frac{3}{4}} \\
\quad - 24a^{\frac{3}{4}}b^{\frac{1}{4}} - 12a^{\frac{1}{2}}b^{\frac{1}{2}} - 15a^{\frac{1}{4}}b^{\frac{3}{4}} - 27b \\
\hline
16a + 8a^{\frac{3}{4}}b^{\frac{1}{4}} + 10a^{\frac{1}{2}}b^{\frac{1}{2}} + 18a^{\frac{1}{4}}b^{\frac{3}{4}} - 24a^{\frac{3}{4}}b^{\frac{1}{4}}, \text{ etc.}
\end{array}$$

9.

$$\begin{array}{l}
x^{\frac{1}{3}} + a^{\frac{1}{3}} \\
x^{\frac{1}{3}} + a^{\frac{1}{3}} \\
\hline
x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} \\
\quad + a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \\
\hline
x^{\frac{2}{3}} + 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}
\end{array}$$

10.

$$\begin{array}{l}
x^{\frac{1}{3}} - a^{\frac{1}{3}} \\
x^{\frac{1}{3}} - a^{\frac{1}{3}} \\
\hline
x^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} \\
\quad - a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \\
\hline
x^{\frac{2}{3}} - 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}
\end{array}$$

11.

$$\begin{array}{l}
x^{\frac{2}{3}} + y^{\frac{2}{3}} \\
x^{\frac{2}{3}} + y^{\frac{2}{3}} \\
\hline
x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} \\
\quad + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}} \\
\hline
x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}
\end{array}$$

12.

$$\begin{array}{l}
a + b^{\frac{1}{3}} \\
a + b^{\frac{1}{3}} \\
\hline
a^2 + ab^{\frac{1}{3}} \\
\quad + ab^{\frac{1}{3}} + b^{\frac{2}{3}} \\
\hline
a^2 + 2ab^{\frac{1}{3}} + b^{\frac{2}{3}}
\end{array}$$

13.

$$\begin{array}{l}
x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 3 \\
x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 3 \\
\hline
x - 2x^{\frac{3}{4}} + 3x^{\frac{1}{2}} \\
\quad - 2x^{\frac{3}{4}} + 4x^{\frac{1}{2}} - 6x^{\frac{1}{4}} \\
\qquad + 3x^{\frac{1}{2}} - 6x^{\frac{1}{4}} + 9 \\
\hline
x - 4x^{\frac{3}{4}} + 10x^{\frac{1}{2}} - 12x^{\frac{1}{4}} + 9
\end{array}$$

14.

$$\begin{array}{l}
2x^{\frac{2}{5}} + 3x^{\frac{1}{5}} + 4 \\
2x^{\frac{2}{5}} + 3x^{\frac{1}{5}} + 4 \\
\hline
4x^{\frac{4}{5}} + 6x^{\frac{3}{5}} + 8x^{\frac{2}{5}} \\
\quad + 6x^{\frac{3}{5}} + 9x^{\frac{2}{5}} + 12x^{\frac{1}{5}} \\
\qquad + 8x^{\frac{2}{5}} + 12x^{\frac{1}{5}} + 16 \\
\hline
4x^{\frac{4}{5}} + 12x^{\frac{3}{5}} + 25x^{\frac{2}{5}} + 24x^{\frac{1}{5}} + 16
\end{array}$$

15.
$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}} \\
\underline{x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}}} \\
x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{1}{3}} \\
\quad - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} \\
\qquad\qquad + x^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}} \\
\hline
x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}}z^{\frac{1}{3}} - 2y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}
\end{array}$$

16.
$$\begin{array}{l}
x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}} \\
\underline{x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}}} \\
x^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{\frac{1}{4}} - x^{\frac{1}{4}}z^{\frac{1}{4}} \\
\quad + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} \\
\qquad\qquad - x^{\frac{1}{4}}z^{\frac{1}{4}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}} \\
\hline
x^{\frac{1}{2}} + 4x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2x^{\frac{1}{4}}z^{\frac{1}{4}} - 4y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}
\end{array}$$

CIV.

1.
$$\begin{array}{l}
x^{\frac{1}{2}} - y^{\frac{1}{2}})\, x - y\, (x^{\frac{1}{2}} + y^{\frac{1}{2}} \\
\qquad\quad \underline{x - x^{\frac{1}{2}}y^{\frac{1}{2}}} \\
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} - y \\
\qquad\qquad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} - y}
\end{array}$$

2.
$$\begin{array}{l}
a^{\frac{1}{2}} + b^{\frac{1}{2}})\, a - b\, (a^{\frac{1}{2}} - b^{\frac{1}{2}} \\
\qquad\quad \underline{a + a^{\frac{1}{2}}b^{\frac{1}{2}}} \\
\qquad\quad - a^{\frac{1}{2}}b^{\frac{1}{2}} - b \\
\qquad\quad \underline{- a^{\frac{1}{2}}b^{\frac{1}{2}} - b}
\end{array}$$

3.
$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}})\, x - y\, (x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\qquad\quad \underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}} \\
\qquad\qquad x^{\frac{2}{3}}y^{\frac{1}{3}} - y \\
\qquad\qquad \underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}}} \\
\qquad\qquad\qquad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\
\qquad\qquad\qquad \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}$$

4.
$$\begin{array}{l}
a^{\frac{1}{3}} + b^{\frac{1}{3}})\, a + b\, (a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \\
\qquad\quad \underline{a + a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
\qquad\quad - a^{\frac{2}{3}}b^{\frac{1}{3}} + b \\
\qquad\quad \underline{- a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}}} \\
\qquad\qquad\qquad a^{\frac{1}{3}}b^{\frac{2}{3}} + b \\
\qquad\qquad\qquad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}} + b}
\end{array}$$

5. $x^{\frac{1}{3}}+y^{\frac{1}{3}})\,x+y\,(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}$, etc.

$$\begin{array}{l} x+x^{\frac{2}{3}}y^{\frac{1}{3}} \\ \hline \quad -x^{\frac{2}{3}}y^{\frac{1}{3}}+y, \text{ etc.} \end{array}$$

6. $m^{\frac{1}{3}}-n^{\frac{1}{3}})\,m-n\,(m^{\frac{2}{3}}+m^{\frac{1}{3}}n^{\frac{1}{3}}$, etc.

$$\begin{array}{l} m-m^{\frac{2}{3}}n^{\frac{1}{3}} \\ \hline \quad m^{\frac{2}{3}}n^{\frac{1}{3}}-n, \text{ etc.} \end{array}$$

7. $x^{\frac{1}{4}}-3y^{\frac{1}{4}})\,x-81y\,(x^{\frac{3}{4}}+3x^{\frac{1}{2}}y^{\frac{1}{4}}$, etc.

$$\begin{array}{l} x-3x^{\frac{3}{4}}y^{\frac{1}{4}} \\ \hline \quad 3x^{\frac{3}{4}}y^{\frac{1}{4}}-81y, \text{ etc.} \end{array}$$

8. $3a^{\frac{1}{4}}-2b^{\frac{1}{4}})\,81a-16b\,(27a^{\frac{3}{4}}+18a^{\frac{1}{2}}b^{\frac{1}{4}}$, etc.

$$\begin{array}{l} 81a-54a^{\frac{3}{4}}b^{\frac{1}{4}} \\ \hline \quad 54a^{\frac{3}{4}}b^{\frac{1}{4}}-16b, \text{ etc.} \end{array}$$

9. $a^{\frac{1}{2}}+x^{\frac{1}{2}})\,a-x\,(a^{\frac{1}{2}}-x^{\frac{1}{2}}$

$$\begin{array}{l} a+a^{\frac{1}{2}}x^{\frac{1}{2}} \\ \hline \quad -a^{\frac{1}{2}}x^{\frac{1}{2}}-x \\ \quad -a^{\frac{1}{2}}x^{\frac{1}{2}}-x \\ \hline \end{array}$$

10. $m^{\frac{1}{5}}-3)\,m-243\,(m^{\frac{4}{5}}+3m^{\frac{3}{5}}$, etc.

$$\begin{array}{l} m-3m^{\frac{4}{5}} \\ \hline \quad 3m^{\frac{4}{5}}-243 \\ \quad 3m^{\frac{4}{5}}-9m^{\frac{3}{5}}, \text{ etc.} \\ \hline \end{array}$$

11. $x^{\frac{1}{2}}+7)\,x+17x^{\frac{1}{2}}+70\,(x^{\frac{1}{2}}+10$

$$\begin{array}{l} x+7x^{\frac{1}{2}} \\ \hline \quad 10x^{\frac{1}{2}}+70 \\ \quad 10x^{\frac{1}{2}}+70 \\ \hline \end{array}$$

12. $x^{\frac{1}{3}}-3)\,x^{\frac{2}{3}}+x^{\frac{1}{3}}-12\,(x^{\frac{1}{3}}+4$

$$\begin{array}{l} x^{\frac{2}{3}}-3x^{\frac{1}{3}} \\ \hline \quad 4x^{\frac{1}{3}}-12 \\ \quad 4x^{\frac{1}{3}}-12 \\ \hline \end{array}$$

13. $b^{\frac{1}{4}}-1)\,-b^{\frac{5}{4}}+3b-3b^{\frac{3}{4}}+b^{\frac{1}{2}}\,(-b+2b^{\frac{3}{4}}-b^{\frac{1}{2}}$

$$\begin{array}{l} -b^{\frac{5}{4}}+b \\ \hline \quad 2b-3b^{\frac{3}{4}} \\ \quad 2b-2b^{\frac{3}{4}} \\ \hline \qquad -b^{\frac{3}{4}}+b^{\frac{1}{2}} \\ \qquad -b^{\frac{3}{4}}+b^{\frac{1}{2}} \\ \hline \end{array}$$

14. $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}})\,x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}\,(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$

$$\begin{array}{l}
\underline{x+x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{2}{3}}z^{\frac{1}{3}}} \\
-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{2}{3}}z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+y+z \\
\underline{-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}} \\
x^{\frac{1}{3}}y^{\frac{2}{3}}+y-x^{\frac{2}{3}}z^{\frac{1}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+z \\
\underline{x^{\frac{1}{3}}y^{\frac{2}{3}}+y+y^{\frac{2}{3}}z^{\frac{1}{3}}} \\
-x^{\frac{2}{3}}z^{\frac{1}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}+z \\
\underline{-x^{\frac{2}{3}}z^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-x^{\frac{1}{3}}z^{\frac{2}{3}}} \\
-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}+x^{\frac{1}{3}}z^{\frac{2}{3}}+z \\
\underline{-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{2}{3}}} \\
x^{\frac{1}{3}}z^{\frac{2}{3}}+y^{\frac{1}{3}}z^{\frac{2}{3}}+z \\
\underline{x^{\frac{1}{3}}z^{\frac{2}{3}}+y^{\frac{1}{3}}z^{\frac{2}{3}}+z}
\end{array}$$

15. $x^{\frac{1}{3}}+4)\,x-5x^{\frac{2}{3}}-46x^{\frac{1}{3}}-40\,(x^{\frac{2}{3}}-9x^{\frac{1}{3}}-10$

$$\begin{array}{l}
\underline{x+4x^{\frac{2}{3}}} \\
-9x^{\frac{2}{3}}-46x^{\frac{1}{3}} \\
\underline{-9x^{\frac{2}{3}}-36x^{\frac{1}{3}}} \\
-10x^{\frac{1}{3}}-40 \\
\underline{-10x^{\frac{1}{3}}-40}
\end{array}$$

16. $m^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{1}{4}}+n^{\frac{1}{2}})\,m+m^{\frac{1}{2}}n^{\frac{1}{2}}+n\,(m^{\frac{1}{2}}+m^{\frac{1}{4}}n^{\frac{1}{4}}+n^{\frac{1}{2}}$

$$\begin{array}{l}
\underline{m-m^{\frac{3}{4}}n^{\frac{1}{4}}+m^{\frac{1}{2}}n^{\frac{1}{2}}} \\
m^{\frac{3}{4}}n^{\frac{1}{4}}+n \\
\underline{m^{\frac{3}{4}}n^{\frac{1}{4}}-m^{\frac{1}{2}}n^{\frac{1}{2}}+m^{\frac{1}{4}}n^{\frac{3}{4}}} \\
m^{\frac{1}{2}}n^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{3}{4}}+n \\
\underline{m^{\frac{1}{2}}n^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{3}{4}}+n}
\end{array}$$

17. $p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1)\,p - 4p^{\frac{3}{4}} + 6p^{\frac{1}{2}} - 4p^{\frac{1}{4}} + 1\,(p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1$

$$\begin{array}{l}
\underline{p - 2p^{\frac{3}{4}} + p^{\frac{1}{2}}} \\
\quad - 2p^{\frac{3}{4}} + 5p^{\frac{1}{2}} - 4p^{\frac{1}{4}} \\
\quad \underline{- 2p^{\frac{3}{4}} + 4p^{\frac{1}{2}} - 2p^{\frac{1}{4}}} \\
\qquad\qquad p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1 \\
\qquad\qquad \underline{p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1}
\end{array}$$

18. $2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}})\,2x + x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - 4y^{\frac{1}{2}}z^{\frac{1}{2}} - x^{\frac{1}{2}}z^{\frac{1}{2}} - z\,(x^{\frac{1}{2}} - y^{\frac{1}{2}} - z^{\frac{1}{2}}$

$$\begin{array}{l}
\underline{2x + 3x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}z^{\frac{1}{2}}} \\
\quad - 2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - 4y^{\frac{1}{2}}z^{\frac{1}{2}} - 2x^{\frac{1}{2}}z^{\frac{1}{2}} - z \\
\quad \underline{- 2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - y^{\frac{1}{2}}z^{\frac{1}{2}} \qquad\qquad\qquad\quad} \\
\qquad\qquad - 2x^{\frac{1}{2}}z^{\frac{1}{2}} - 3y^{\frac{1}{2}}z^{\frac{1}{2}} - z \\
\qquad\qquad \underline{- 2x^{\frac{1}{2}}z^{\frac{1}{2}} - 3y^{\frac{1}{2}}z^{\frac{1}{2}} - z}
\end{array}$$

19. $x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}})\,x + y\,(x^{\frac{1}{5}} + y^{\frac{1}{5}}$

$$\begin{array}{l}
\underline{x - x^{\frac{4}{5}}y^{\frac{1}{5}} + x^{\frac{3}{5}}y^{\frac{2}{5}} - x^{\frac{2}{5}}y^{\frac{3}{5}} + x^{\frac{1}{5}}y^{\frac{4}{5}}} \\
\quad x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y \\
\quad \underline{x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y}
\end{array}$$

CV.

1.
$$\begin{array}{lcr}
a^{-1} + b^{-1} & & \\
\underline{a^{-1} - b^{-1}} & & \\
a^{-2} + a^{-1}b^{-1} & & \\
\quad - a^{-1}b^{-1} - b^{-2} & & \\
\hline
a^{-2} & & - b^{-2}
\end{array}$$

2.
$$\begin{array}{lcr}
x^{-3} + b^{-2} & & \\
\underline{x^{-3} - b^{-2}} & & \\
x^{-6} + x^{-3}b^{-2} & & \\
\quad - x^{-3}b^{-2} - b^{-4} & & \\
\hline
x^{-6} & & - b^{-4}
\end{array}$$

3.
$$\begin{array}{l}
x^3+x+x^{-1}+x^{-3}\\
x-x^{-1}\\
\hline
x^4+x^2+1+x^{-2}\\
\qquad -x^2-1-x^{-2}-x^{-4}\\
\hline
x^4 \qquad\qquad\qquad -x^{-4}
\end{array}$$

4.
$$\begin{array}{l}
x^2-1+x^{-2}\\
x^2+1+x^{-2}\\
\hline
x^4-x^2+1\\
\quad +x^2-1+x^{-2}\\
\qquad\quad +1-x^{-2}+x^{-4}\\
\hline
x^4 \qquad +1+x^{-4}
\end{array}$$

5.
$$\begin{array}{l}
a^{-2}+b^{-2}\\
a^{-2}-b^{-2}\\
\hline
a^{-4}+a^{-2}b^{-2}\\
\qquad -a^{-2}b^{-1}-b^{-4}\\
\hline
a^{-4} \qquad\qquad -b^{-4}
\end{array}$$

6.
$$\begin{array}{l}
a^{-1}-b^{-1}+c^{-1}\\
a^{-1}+b^{-1}+c^{-1}\\
\hline
a^{-2}-a^{-1}b^{-1}+a^{-1}c^{-1}\\
\qquad +a^{-1}b^{-1}-b^{-2}+b^{-1}c^{-1}\\
\qquad\qquad\qquad +a^{-1}c^{-1}-b^{-1}c^{-1}+c^{-2}\\
\hline
a^{-2}+2a^{-1}c^{-1}-b^{-2}+c^{-2}
\end{array}$$

7.
$$\begin{array}{l}
1+ab^{-1}+a^2b^{-2}\\
1-ab^{-1}+a^2b^{-2}\\
\hline
1+ab^{-1}+a^2b^{-2}\\
\quad -ab^{-1}-a^2b^{-2}-a^3b^{-3}\\
\qquad\qquad +a^2b^{-2}+a^3b^{-3}+a^4b^{-4}\\
\hline
1 \qquad\qquad +a^2b^{-2}+a^4b^{-4}
\end{array}$$

8.
$$\begin{array}{l}
a^2b^{-2}+2+a^{-2}b^2\\
a^2b^{-2}-2-a^{-2}b^2\\
\hline
a^4b^{-4}+2a^2b^{-2}+1\\
\qquad -2a^2b^{-2}-4-2a^{-2}b^2\\
\qquad\qquad\qquad -1-2a^{-2}b^2-a^{-4}b^4\\
\hline
a^4b^{-4}\cdot\ 4\ -a^{-4}b^4-4a^{-2}b^2
\end{array}$$

9.
$$\begin{array}{l}
4x^{-3}+3x^{-2}+2x^{-1}+1 \\
x^{-2}-x^{-1}+1 \\
\hline
4x^{-5}+3x^{-4}+2x^{-3}+x^{-2} \\
\quad -4x^{-4}-3x^{-3}-2x^{-2}-x^{-1} \\
\qquad +4x^{-3}+3x^{-2}+2x^{-1}+1 \\
\hline
4x^{-5}-x^{-4}+3x^{-3}+2x^{-2}+x^{-1}+1
\end{array}$$

10.
$$\begin{array}{l}
\frac{5}{2}x^{-2}+3x^{-1}-\frac{7}{3} \\
2x^{-2}-x^{-1}-\frac{1}{2} \\
\hline
5x^{-4}+6x^{-3}-\frac{14}{3}x^{-2} \\
\quad -\frac{5}{2}x^{-3}-3x^{-2}+\frac{7}{3}x^{-1} \\
\qquad -\frac{5}{4}x^{-2}-\frac{3}{2}x^{-1}+\frac{7}{6} \\
\hline
5x^{-4}+\frac{7}{2}x^{-3}-\frac{107}{12}x^{-2}+\frac{5}{6}x^{-1}+\frac{7}{6}
\end{array}$$

CVI.

1.
$$\begin{array}{l}
x+x^{-1})x^2-x^{-2}(x-x^{-1} \\
\quad x^2+1 \\
\hline
\quad -1-x^{-2} \\
\quad -1-x^{-2} \\
\hline
\end{array}$$

2.
$$\begin{array}{l}
a-b^{-1})a^2-b^{-2}(a+b^{-1} \\
\quad a^2-ab^{-1} \\
\hline
\quad ab^{-1}-b^{-2} \\
\quad ab^{-1}-b^{-2} \\
\hline
\end{array}$$

3.
$$\begin{array}{l}
m+n^{-1})m^3+n^{-3}(m^2-mn^{-1}+n^{-2} \\
\quad m^3+m^2n^{-1} \\
\hline
\quad -m^2n^{-1}+n^{-3} \\
\quad -m^2n^{-1}-mn^{-2} \\
\hline
\qquad mn^{-2}+n^{-3} \\
\qquad mn^{-2}+n^{-3} \\
\hline
\end{array}$$

4. $c-d^{-1})c^5-d^{-5}(c^4+c^3d^{-1}+c^2d^{-2}$, etc.

$$\begin{array}{l}
\underline{c^5-c^4d^{-1}} \\
\quad c^4d^{-1}-d^{-5} \\
\quad \underline{c^4d^{-1}-c^3d^{-2}} \\
\qquad c^3d^{-2}-d^{-5} \\
\qquad \underline{c^3d^{-2}-c^2d^{-3}}, \text{ etc.}
\end{array}$$

5. $xy^{-1}+x^{-1}y)x^2y^{-2}+2+x^{-2}y^2(xy^{-1}+x^{-1}y$

$$\begin{array}{l}
\underline{x^2y^{-2}+1} \\
\qquad 1+x^{-2}y^2 \\
\qquad \underline{1+x^{-2}y^2}
\end{array}$$

6. $a^{-2}-a^{-1}b^{-1}+b^{-2})a^{-4}+a^{-2}b^{-2}+b^{-4}(a^{-2}+a^{-1}b^{-1}+b^{-2}$

$$\begin{array}{l}
\underline{a^{-4}-a^{-3}b^{-1}+a^{-2}b^{-2}} \\
\quad a^{-3}b^{-1}+b^{-4} \\
\quad \underline{a^{-3}b^{-1}-a^{-2}b^{-2}+a^{-1}b^{-3}} \\
\qquad a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4} \\
\qquad \underline{a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}}
\end{array}$$

7. $xy^{-1}-x^{-1}y)x^3y^{-3}-3xy^{-1}+3x^{-1}y-x^{-3}y^3(x^2y^{-2}-2+x^{-2}y^2$

$$\begin{array}{l}
\underline{x^3y^{-3}-\ xy^{-1}} \\
\quad -2xy^{-1}+3x^{-1}y \\
\quad \underline{-2xy^{-1}+2x^{-1}y} \\
\qquad x^{-1}y-x^{-3}y^3 \\
\qquad \underline{x^{-1}y-x^{-3}y^3}
\end{array}$$

8. $$\frac{x^{-2}}{2}-x^{-1}+3\Big)\frac{3x^{-5}}{4}-4x^{-4}+\frac{77x^{-3}}{8}-\frac{43x^{-2}}{4}-\frac{33x^{-1}}{4}+27\Big(\frac{3x^{-3}}{2}-5x^{-2}+\frac{x^{-1}}{4}+9$$

$$\frac{3x^{-5}}{4}-\frac{3x^{-4}}{2}+\frac{9x^{-3}}{2}$$

$$-\frac{5x^{-4}}{2}+\frac{41x^{-3}}{8}-\frac{43x^{-2}}{4}$$

$$-\frac{5x^{-4}}{2}+5x^{-3}-15x^{-2}$$

$$\frac{x^{-3}}{8}+\frac{17x^{-2}}{4}-\frac{33x^{-1}}{4}$$

$$\frac{x^{-3}}{8}-\frac{x^{-2}}{4}+\frac{3x^{-1}}{4}$$

$$\frac{9x^{-2}}{2}-9x^{-1}+27$$

$$\frac{9x^{-2}}{2}-9x^{-1}+27$$

9. $$ab^{-1}+a^{-1}b)\,a^3b^{-3}+a^{-3}b^3\,(a^2b^{-2}-1+a^{-2}b^2$$

$$a^3b^{-3}+ab^{-1}$$

$$-ab^{-1}+a^{-3}b^3$$

$$-ab^{-1}-a^{-1}b$$

$$a^{-1}b+a^{-3}b^3$$

$$a^{-1}b+a^{-3}b^3$$

10. $$\begin{array}{l}
a^{-1}+b^{-1}+c^{-1})\,a^{-3}+b^{-3}+c^{-3}-3a^{-1}b^{-1}c^{-1}\,(a^{-2}-a^{-1}b^{-1}-a^{-1}c^{-1}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +b^{-2}-b^{-1}c^{-1}+c^{-2}\\
\underline{a^{-3}+a^{-2}b^{-1}+a^{-2}c^{-1}}\\
\quad -a^{-2}b^{-1}-a^{-2}c^{-1}-3a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3}\\
\quad \underline{-a^{-2}b^{-1}-a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}}\\
\qquad -a^{-2}c^{-1}+a^{-1}b^{-2}-2a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3}\\
\qquad \underline{-a^{-2}c^{-1}-a^{-1}b^{-1}c^{-1}-a^{-1}c^{-2}}\\
\qquad\quad a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}+b^{-3}+c^{-3}\\
\qquad\quad \underline{a^{-1}b^{-2}+b^{-3}+b^{-2}c^{-1}}\\
\qquad\qquad -a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}-b^{-2}c^{-1}+c^{-3}\\
\qquad\qquad \underline{-a^{-1}b^{-1}c^{-1}-b^{-2}c^{-1}-b^{-1}c^{-2}}\\
\qquad\qquad\qquad a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}\\
\qquad\qquad\qquad \underline{a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}}
\end{array}$$

CVII.

1. $$\begin{array}{l}
x^{\frac{2}{3}}+2x^{\frac{1}{2}}y^{\frac{1}{2}}+2y)\,x^{\frac{4}{3}}-4xy+4x^{\frac{2}{3}}y+4y^2\,(x^{\frac{2}{3}}-2x^{\frac{1}{2}}y^{\frac{1}{2}}+2y\\
\qquad\qquad \underline{x^{\frac{4}{3}}+2x^{\frac{7}{6}}y^{\frac{1}{2}}+2x^{\frac{2}{3}}y}\\
\qquad\qquad\quad -2x^{\frac{7}{6}}y^{\frac{1}{2}}-4xy+2x^{\frac{2}{3}}y\\
\qquad\qquad\quad \underline{-2x^{\frac{7}{6}}y^{\frac{1}{2}}-4xy-4x^{\frac{1}{2}}y^{\frac{3}{2}}}\\
\qquad\qquad\qquad 2x^{\frac{2}{3}}y+4x^{\frac{1}{2}}y^{\frac{3}{2}}+4y^2\\
\qquad\qquad\qquad \underline{2x^{\frac{2}{3}}y+4x^{\frac{1}{2}}y^{\frac{3}{2}}+4y^2}
\end{array}$$

2. $\{x^{15ab}\times x^{12}\}^{\frac{1}{3a-2}}=x^{\frac{15ab+12}{3a-2}}$.

3. $(x^{10b+18a})^{\frac{1}{3a-2}}=x^{\frac{10b+18a}{3a-2}}$.

4. $\left\{\dfrac{x^2+a^2-x^2+a^2}{x^4-a^4}-\dfrac{a(x-a-x-a)}{x^4-a^4}\right\}=\left(\dfrac{4a^2}{x^4-a^4}\right)^{\frac{1}{2}}$, etc.

5.
$$
\begin{array}{l}
\frac{7}{3}x^{-2}+4x^{-1}-\frac{2}{7} \\
3x^{-2}-2x^{-1}-\frac{1}{2} \\
\hline
7x^{-4}+12x^{-3}-\frac{6}{7}x^{-2} \\
\qquad -\frac{14}{3}x^{-3}-8x^{-2}+\frac{4}{7}x^{-1} \\
\qquad\qquad -\frac{7}{6}x^{-2}-2x^{-1}+\frac{1}{7} \\
\hline
7x^{-4}+\frac{22}{3}x^{-3}-\frac{421}{42}x^{-2}-\frac{10}{7}x^{-1}+\frac{1}{7}
\end{array}
$$

6. $\dfrac{x^{a+b+a-b+c-2a}}{x^{c-a}}=x^{c-c+a}=x^{a}.$

7.
$$
\begin{array}{l}
x^n+y^n)\,x^{2n}-y^{2n}\,(x^n-y^n \\
\qquad x^{2n}+x^ny^n \\
\qquad \overline{\qquad -x^ny^n-y^{2n}} \\
\qquad\qquad -x^ny^n-y^{2n} \\
\hline
\end{array}
$$

8.
$$
\begin{array}{l}
a^{\frac{3}{2}}+3ab^{\frac{5}{6}}+3a^{\frac{1}{2}}b^{\frac{5}{3}}+b^{\frac{5}{2}} \\
a^{\frac{1}{2}}-b^{\frac{5}{6}} \\
\hline
a^2+3a^{\frac{3}{2}}b^{\frac{5}{6}}+3ab^{\frac{5}{3}}+a^{\frac{1}{2}}b^{\frac{5}{2}} \\
\quad -\ a^{\frac{3}{2}}b^{\frac{5}{6}}-3ab^{\frac{5}{3}}-3a^{\frac{1}{2}}b^{\frac{5}{2}}-b^{\frac{10}{3}} \\
\hline
a^2+2a^{\frac{3}{2}}b^{\frac{5}{6}} \qquad\qquad -2a^{\frac{1}{2}}b^{\frac{5}{2}}-b^{\frac{10}{3}}
\end{array}
$$

9.
$$
\begin{array}{l}
a^{\frac{1}{3}}-b^{\frac{1}{3}})\,a-b\,(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}},\text{ etc.} \\
\qquad a-a^{\frac{2}{3}}b^{\frac{1}{3}} \\
\hline
\qquad\qquad a^{\frac{2}{3}}b^{\frac{1}{3}}-b,\text{ etc.}
\end{array}
$$

10. $(a^2)^m = a^{2+2+\cdots \; m \text{ terms}} = a^{2m} = a^m \times a^m = (a^m)^2$.

11. $a^{m^n} = a^{mn}$; $\therefore m^n = mn$; $m^{n-1} = n$; $m = n^{\frac{1}{n-1}}$.

12. $x^{a+b+c+a+b-c+a-b+c+b+c-a} = x^{2a+2b+2c}$.

13. $(x^p)^p \div (x^p)^{p-q} = x^{p^2} \div x^{p^2-pq} = x^{pq}$.

14. $4a^x \div \dfrac{1}{4a^x} = 4a^x \times 4a^x = 16a^{2x}$.

15. $[\{a^{mn}\}^p] \div [\{a^{mn}\}^{-p}] = a^{mnp} \div a^{-mnp} = a^{2mnp}$.

16. $2a^{2m} + 2a^m b^p - 4a^m c^n - 3a^m b - 3b^{p+1} + 6bc^n$.

17. $a^{m-n+n-m}$. $b^{n-p+p-n}$. $c = 1 \times 1 \times c = c$.

18. $$\frac{a^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}})}{(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})} = \frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}.$$

19. $(x^{\frac{1}{3}} + x^{\frac{1}{6}} + 1)(x^{\frac{1}{3}} - x^{\frac{1}{6}} + 1) = x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$
$(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1) = x^{\frac{4}{3}} + x^{\frac{2}{3}} + 1$

20.
$$\begin{array}{l}
a^m - ba^{m-1}x + ca^{m-2}x^2 \\
a^n + ba^{n-1}x - ca^{n-2}x^2 \\
\hline
a^{m+n} - ba^{m+n-1}x + ca^{m+n-2}x^2 \\
\qquad + ba^{m+n-1}x - b^2a^{m+n-2}x^2 + bca^{m+n-3}x^3 \\
\qquad\qquad - ca^{m+n-2}x^2 + bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4 \\
\hline
a^{m+n} - b^2a^{m+n-2}x^2 + 2bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4
\end{array}$$

21.
$$\begin{array}{l}
x^{pq-p} + y^{pq-q})\, x^{2pq-2p} - y^{2pq-2q}\, (x^{pq-p} - y^{pq-q} \\
\qquad\qquad x^{2pq-2p} + x^{pq-p}y^{pq-q} \\
\qquad\qquad \overline{\qquad\qquad - x^{pq-p}y^{pq-q} - y^{2pq-2q}} \\
\qquad\qquad\qquad - x^{pq-p}y^{pq-q} - y^{2pq-2q} \\
\hline
\end{array}$$

22. $\{(a^m)^{\frac{m^2-1}{m}}\}^{\frac{1}{m+1}} = a^{\frac{m^2-1}{m+1}} = a^{m-1}$.

23. $x^{3r} + x^{2r}y^p + x^r y^{2p} + y^{3p}$

$$\begin{array}{l}
x^{3r} + x^{2r}y^p + x^r y^{2p} + y^{3p} \\
x^r - y^p \\
\hline
x^{4r} + x^{3r}y^p + x^{2r}y^{2p} + x^r y^{3p} \\
\qquad - x^{3r}y^p - x^{2r}y^{2p} - x^r y^{3p} - y^{4p} \\
\hline
x^{4r} \qquad\qquad\qquad\qquad\qquad\qquad - y^{4p}
\end{array}$$

24. $\sqrt[4]{625} = \sqrt{25} = 5$; and $\dfrac{1}{12^2} = \dfrac{1}{144}$.

25.
$$\begin{array}{l}
x^{mn-n} - y^{mn-m} \\
x^n - y^m \\
\hline
x^{mn} - x^n y^{mn-m} - x^{mn-n} y^m + y^{mn}
\end{array}$$

26.
$$\begin{array}{l}
x^{\frac{3}{4}} + 3x^{\frac{1}{2}} - 1 \\
x^{\frac{1}{4}} - 2x^{-\frac{1}{4}} \\
\hline
x + 3x^{\frac{3}{4}} - x^{\frac{1}{4}} \\
\quad - 2x^{\frac{1}{2}} - 6x^{\frac{1}{4}} + 2x^{-\frac{1}{4}} \\
\hline
x + 3x^{\frac{3}{4}} - 2x^{\frac{1}{2}} - 7x^{\frac{1}{4}} + 2x^{-\frac{1}{4}}
\end{array}$$

CVIII.

1. $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$; $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; $\sqrt[6]{x^3}$ and $\sqrt[6]{y^2}$.

2. $4^{\frac{1}{3}}$ and $2^{\frac{1}{5}}$; $4^{\frac{5}{15}}$ and $2^{\frac{3}{15}}$; $\sqrt[15]{4^5}$ and $\sqrt[15]{2^3}$, etc.

3. $18^{\frac{1}{2}}$ and $50^{\frac{1}{3}}$; $18^{\frac{3}{6}}$ and $50^{\frac{2}{6}}$; $\sqrt[6]{18^3}$ and $\sqrt[6]{50^2}$, etc.

4. $2^{\frac{1}{m}}$ and $2^{\frac{1}{n}}$; $2^{\frac{n}{mn}}$ and $2^{\frac{m}{mn}}$; $\sqrt[mn]{2^n}$ and $\sqrt[mn]{2^m}$.

5. $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$; $a^{\frac{n}{mn}}$ and $b^{\frac{m}{mn}}$; $\sqrt[mn]{a^n}$ and $\sqrt[mn]{b^m}$.

6. $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$; $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$, etc.

CIX.

1. $\sqrt{24}=\sqrt{(4\times 6)}=2\sqrt{6}$.

2. $\sqrt{50}=\sqrt{(25\times 2)}=5\sqrt{2}$.

3. $\sqrt{(4a^3)}=2a\sqrt{a}$.

4. $\sqrt{(125a^4d^3)}=5a^2d\sqrt{(5d)}$.

5. $\sqrt{(16\times 2yz^3)}=4z\sqrt{(2yz)}$.

6. $\sqrt{(100\times 10a)}=10\sqrt{(10a)}$.

7. $\sqrt{(144\times 5c^2)}=12c\sqrt{5}$.

8. $7.\sqrt{(36\times 11x)}=42\sqrt{(11x)}$.

9. $18.\sqrt{\left(\frac{1}{9}\times\frac{5}{3}x^3\right)}=6x\sqrt{\frac{5x}{3}}$.

10. $a\sqrt{\left(a^2\times\frac{a}{b}\right)}=a^2\sqrt{\frac{a}{b}}$.

11. $\sqrt{\{a(a^2+2ax+x^2)\}}=(a+x)\sqrt{a}$.

12. $\sqrt{\{x(x^2-2xy+y^2)\}}=(x-y)\sqrt{x}$.

13. $\sqrt{\{2(25a^2-50ab+25b^2)\}}=5(a-b)\sqrt{2}$.

14. $\sqrt{\{7y(9c^4-6c^2y+y^2)\}}=(3c^2-y)\sqrt{(7y)}$.

15. $\sqrt[3]{(27a^6\times 2b^2)}=3a^2\sqrt[3]{(2b^2)}$.

16. $\sqrt[3]{(20xy\times 8x^3y^6)}=2xy^2\sqrt[3]{(20xy)}$.

17. $\sqrt[3]{(27m^9n^9\times 4n)}=3m^3n^3\sqrt[3]{(4n)}$.

18. $\sqrt[3]{(343a^{15}b^{15}\times 4b)}=7a^5b^5\sqrt[3]{(4b)}$.

19. $\sqrt[3]{\{x(x^3+3x^2y+3xy^2+y^3)\}}=(x+y)\sqrt[3]{x}$.

20. $\sqrt[3]{\{a(a^3-3a^2b+3ab^2-b^3)\}}=(a-b)\sqrt[3]{a}$.

CX.

1. $4\sqrt{3}=\sqrt{16}.\sqrt{3}=\sqrt{48}$.

2. $3\sqrt{7}=\sqrt{9}.\sqrt{7}=\sqrt{63}$.

3. $5\sqrt[3]{9}=\sqrt[3]{125}.\sqrt[3]{9}=\sqrt[3]{1125}$.

4. $2\sqrt[4]{6}=\sqrt[4]{16}.\sqrt[4]{6}=\sqrt[4]{96}$.

5. $3\sqrt[3]{\frac{3}{7}}=\sqrt[3]{27}.\sqrt[3]{\frac{3}{7}}=\sqrt[3]{\frac{81}{7}}$.

6. $3\sqrt{a}=\sqrt{9}.\sqrt{a}=\sqrt{(9a)}$.

7. $4a\sqrt{(3x)} = \sqrt{(16a^2)}\sqrt{(3x)} = \sqrt{(48a^2x)}$.

8. $2ax\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(4a^2x^2)}\,.\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(3a^3x)}$.

9. $\sqrt{(m+n)^2}\,.\sqrt{\left(\frac{m-n}{m+n}\right)} = \sqrt{(m^2-n^2)}$.

10. $\sqrt{(a+b)^2}\,.\sqrt{\left(\frac{1}{a^2-b^2}\right)} = \sqrt{\frac{(a+b)^2}{a^2-b^2}} = \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$.

11. $\sqrt{.}\left(\frac{x-y}{x+y}\right)^2\,.\sqrt{\left(\frac{x^2+xy}{x^2-2xy+y^2}\right)} = \sqrt{.}\frac{x^2+xy}{(x+y)^2} = \left(\frac{x}{x+y}\right)^{\frac{1}{2}}$.

CXI.

1. $\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$; $\sqrt[3]{4} = \sqrt[6]{4^2} = \sqrt[6]{16}$, etc.

2. $\sqrt{10} = \sqrt[6]{10^3} = \sqrt[6]{1000}$; $\sqrt[3]{15} = \sqrt[6]{15^2} = \sqrt[6]{225}$, etc.

3. $2\sqrt{3} = \sqrt{(4\times 3)} = \sqrt{12}$; $3\sqrt{2} = \sqrt{(9\times 2)} = \sqrt{18}$, etc.

4. $\sqrt{\frac{3}{5}} = \sqrt[6]{\frac{27}{125}}$; $\sqrt[3]{\frac{14}{15}} = \sqrt[6]{\frac{196}{225}}$, etc.

5. $3\sqrt{7} = \sqrt{(9\times 7)} = \sqrt{63}$; $4\sqrt{3} = \sqrt{(16\times 3)} = \sqrt{48}$, etc.

6. $2\sqrt{87} = \sqrt{(4\times 87)} = \sqrt{348}$; $3\sqrt{33} = \sqrt{(9\times 33)} = \sqrt{297}$, etc.

7. $2\sqrt[3]{22} = \sqrt[3]{176} = \sqrt[6]{30976}$; $3\sqrt[3]{7} = \sqrt[3]{189} = \sqrt[6]{35721}$;
$4\sqrt{2} = \sqrt{32} = \sqrt[6]{32768}$, etc.

8. $3\sqrt{19} = \sqrt{171} = \sqrt[6]{5000211}$; $5\sqrt[3]{18} = \sqrt[3]{2250} = \sqrt[6]{5062500}$;
$3\sqrt[3]{82} = \sqrt[3]{2214} = \sqrt[6]{4901796}$, etc.

9. $2\sqrt[3]{14} = \sqrt[3]{112}$; $5\sqrt[3]{2} = \sqrt[3]{250}$; $3\sqrt[3]{3} = \sqrt[3]{81}$, etc.

10. $\frac{1}{2}\sqrt{2} = \sqrt{\frac{1}{4}}$; $\frac{1}{3}\sqrt{3} = \sqrt{\frac{1}{9}}$; $\frac{1}{4}\sqrt{4} = \sqrt{\frac{1}{16}}$, etc.

CXII.

1. $3\sqrt{3}+8\sqrt{3}+18\sqrt{3}=29\sqrt{3}$.

2. $30\sqrt{10}+20\sqrt{2}+144\sqrt{2}=30\sqrt{10}+164\sqrt{2}$.

3. $a^2\sqrt{x}+b^2\sqrt{x}+c^2\sqrt{x}=(a^2+b^2+c^2)\sqrt{x}$.

4. $4\quad 2+7\sqrt[3]{2}+2\sqrt[3]{2}=13\sqrt[3]{2}$.

5. $21\sqrt[3]{2}+6\sqrt[3]{2}+6\sqrt[3]{2}=33\sqrt[3]{2}$.

6. $4\sqrt{6}-3\sqrt{6}=\sqrt{6}$.

7. $9\sqrt{3}-4\sqrt{3}=5\sqrt{3}$.

8. $72\sqrt{2}-24\sqrt{2}=48\sqrt{2}$.

9. $10\sqrt[3]{2}-6\sqrt[3]{2}=4\sqrt[3]{2}$.

10. $21\sqrt[3]{3}-21\sqrt[3]{3}=0$.

11. $\sqrt{6}\times\sqrt{8}=\sqrt{48}=4\sqrt{3}$.

12. $\sqrt{14}\times\sqrt{20}=\sqrt{280}=2\sqrt{70}$.

13. $\sqrt{50}\times\sqrt{200}=\sqrt{10000}=100$.

14. $\sqrt[3]{(3a^2b)}\times\sqrt[3]{(9ab^2)}=\sqrt[3]{(27a^3b^3)}=3ab$.

15. $\sqrt[3]{(12ab)}\times\sqrt[3]{(8a^2b^3)}=\sqrt[3]{(96a^3b^4)}=2ab\sqrt[3]{(12b)}$.

16. $\sqrt{12}\div\sqrt{3}=\sqrt{4}=2$.

17. $\sqrt{18}\div\sqrt{50}=\sqrt{\frac{9}{25}}=\frac{3}{5}$.

18. $\sqrt[3]{(a^2b)}\div\sqrt[3]{(ab^2)}=\sqrt[3]{\left(\frac{a}{b}\right)}$.

19. $\sqrt[4]{(a^3b)}\div\sqrt[4]{(ab^3)}=\sqrt[4]{\left(\frac{a^2}{b^2}\right)}=\sqrt{\left(\frac{a}{b}\right)}$.

20. $\sqrt{(x^2+x^3y)}\div\sqrt{(x+2x^2y+x^3y^2)}=\sqrt{\left(\frac{x+x^2y}{1+2xy+x^2y^2}\right)}=\sqrt{\frac{x}{1+xy}}$.

CXIII.

1. $\sqrt{x}\times\sqrt{y}=\sqrt{(xy)}$.

2. $\sqrt{(x-y)}\times\sqrt{y}=\sqrt{(xy-y^2)}$.

3. $\sqrt{(x+y)}\times\sqrt{(x+y)}=x+y$.

4. $\sqrt{(x-y)}\times\sqrt{(x+y)}=\sqrt{(x^2-y^2)}$.

5. $6\sqrt{x} \times 3\sqrt{x} = 18x$.

6. $7\sqrt{(x+1)} \times 8\sqrt{(x+1)} = 56(x+1)$.

7. $10\sqrt{x} \times 9\sqrt{(x-1)} = 90\sqrt{(x^2-x)}$.

8. $\sqrt{(3x)} \times \sqrt{(4x)} = \sqrt{(12x^2)} = 2x\sqrt{3}$.

9. $\sqrt{x} \times -\sqrt{x} = -x$.

10. $\sqrt{(x-1)} \times -\sqrt{(x-1)} = -(x-1) = 1-x$.

11. $3\sqrt{x} \times -4\sqrt{x} = -12x$.

12. $-2\sqrt{a} \times -3\sqrt{a} = 6a$.

13. $\sqrt{(x-7)} \times -\sqrt{x} = -\sqrt{(x^2-7x)}$.

14. $-2\sqrt{(x+7)} \times -3\sqrt{x} = 6\sqrt{(x^2+7x)}$.

15. $-4\sqrt{(a^2-1)} \times -2\sqrt{(a^2-1)} = 8(a^2-1)$.

16. $2\sqrt{(a^2-2a+3)} \times -3\sqrt{(a^2-2a+3)} = -6(a^2-2a+3) =$
$-6a^2+12a-18$.

CXIV.

1.
$$\begin{array}{l} \sqrt{x}+7 \\ \underline{\sqrt{x}+2} \\ x+7\sqrt{x} \\ \underline{\quad +2\sqrt{x}+14} \\ x+9\sqrt{x}+14 \end{array}$$

2.
$$\begin{array}{l} \sqrt{x}-5 \\ \underline{\sqrt{x}+3} \\ x-5\sqrt{x} \\ \underline{\quad +3\sqrt{x}-15} \\ x-2\sqrt{x}-15 \end{array}$$

3.
$$\begin{array}{l} \sqrt{(a+9)}+3 \\ \underline{\sqrt{(a+9)}-3} \\ a+9+3\sqrt{(a+9)} \\ \underline{\qquad -3\sqrt{(a+9)}-9} \\ a \end{array}$$

4.
$$\begin{array}{l} \sqrt{(a-4)}-7 \\ \underline{\sqrt{(a-4)}+7} \\ a-4-7\sqrt{(a-4)} \\ \underline{\qquad +7\sqrt{(a-4)}-49} \\ a-53 \end{array}$$

5.
$$\begin{array}{l}
3\sqrt{x}-7\\
\sqrt{x}+4\\
\hline
3x-7\sqrt{x}\\
\quad +12\sqrt{x}-28\\
\hline
3x+5\sqrt{x}-28
\end{array}$$

6.
$$\begin{array}{l}
2\sqrt{(x-5)}+4\\
3\sqrt{(x-5)}-6\\
\hline
6(x-5)+12\sqrt{(x-5)}\\
\quad -12\sqrt{(x-5)}-24\\
\hline
6x-54
\end{array}$$

7.
$$\begin{array}{l}
\sqrt{(6+x)}+\sqrt{x}\\
\sqrt{(6+x)}-\sqrt{x}\\
\hline
6+x+\sqrt{(6x+x^2)}\\
\quad -\sqrt{(6x+x^2)}-x\\
\hline
6
\end{array}$$

8.
$$\begin{array}{l}
\sqrt{(3x+1)}+\sqrt{(2x-1)}\\
\sqrt{3x}-\sqrt{(2x-1)}\\
\hline
\sqrt{(9x^2+3x)}+\sqrt{(6x^2-3x)}\\
\quad +\sqrt{(6x^2-x-1)}-(2x-1)\\
\hline
\sqrt{(9x^2+3x)}+\sqrt{(6x^2-3x)}+\sqrt{(6x^2-x-1)}-2x+1
\end{array}$$

9.
$$\begin{array}{l}
\sqrt{a}+\sqrt{(a-x)}\\
\sqrt{x}-\sqrt{(a-x)}\\
\hline
\sqrt{(ax)}+\sqrt{(ax-x^2)}\\
\quad -\sqrt{(a^2-ax)}-(a-x)\\
\hline
\sqrt{(ax)}+\sqrt{(ax-x^2)}-\sqrt{(a^2-ax)}-a+x
\end{array}$$

10.
$$\begin{array}{l}
\sqrt{(3+x)}+\sqrt{x}\\
\sqrt{(3+x)}\\
\hline
3+x+\sqrt{(3x+x^2)}
\end{array}$$

11. $$\begin{array}{l} \sqrt{x}+\sqrt{y}+\sqrt{z} \\ \sqrt{x}-\sqrt{y}+\sqrt{z} \\ \hline x+\sqrt{(xy)}+\sqrt{(xz)} \\ \quad -\sqrt{(xy)}-y-\sqrt{(yz)} \\ \qquad\qquad +\sqrt{(xz)}+\sqrt{(yz)}+z \\ \hline x-y+2\sqrt{(xz)}+z \end{array}$$

12. $$\begin{array}{l} \sqrt{a}+\sqrt{(a-x)}+\sqrt{x} \\ \sqrt{a}-\sqrt{(a-x)}+\sqrt{x} \\ \hline a+\sqrt{(a^2-ax)}+\sqrt{(ax)} \\ \quad -\sqrt{(a^2-ax)}-(a-x)-\sqrt{(ax-x^2)} \\ \qquad\qquad +\sqrt{(ax)}+\sqrt{(ax-x^2)}+x \\ \hline 2x+2\sqrt{(ax)} \end{array}$$

13. $$\begin{array}{l} 21+\sqrt{(x^2-9)} \\ 21+\sqrt{(x^2-9)} \\ \hline 441+21\sqrt{(x^2-9)} \\ \quad +21\sqrt{(x^2-9)}+x^2-9 \\ \hline 432+42\sqrt{(x^2-9)}+x^2 \end{array}$$

14. $$\begin{array}{l} \sqrt{(x+3)}+\sqrt{(x+8)} \\ \sqrt{(x+3)}+\sqrt{(x+8)} \\ \hline x+3+\sqrt{(x^2+11x+24)} \\ \quad +\sqrt{(x^2+11x+24)}+x+8 \\ \hline 2x+11+2\sqrt{(x^2+11x+24)} \end{array}$$

15. $$\begin{array}{l} \sqrt{x}+\sqrt{(x-4)} \\ \sqrt{x}+\sqrt{(x-4)} \\ \hline x+\sqrt{(x^2-4x)} \\ \quad +\sqrt{(x^2-4x)}+x-4 \\ \hline 2x-4+2\sqrt{(x^2-4x)} \end{array}$$

16. $$\begin{array}{l} \sqrt{(x-6)}+\sqrt{x} \\ \sqrt{(x-6)}+\sqrt{x} \\ \hline x-6+\sqrt{(x^2-6x)} \\ \quad +\sqrt{(x^2-6x)}+x \\ \hline 2x-6+2\sqrt{(x^2-6x)} \end{array}$$

17.
$$\begin{array}{l} 2\sqrt{x}-3 \\ 2\sqrt{x}-3 \\ \hline 4x-6\sqrt{x} \\ \qquad -6\sqrt{x}+9 \\ \hline 4x-12\sqrt{x}+9 \end{array}$$

18.
$$\begin{array}{l} \sqrt{(x+y)}-\sqrt{(x-y)} \\ \sqrt{(x+y)}-\sqrt{(x-y)} \\ \hline x+y-\sqrt{(x^2-y^2)} \\ \qquad -\sqrt{(x^2-y^2)}+x-y \\ \hline 2x-2\sqrt{(x^2-y^2)} \end{array}$$

19.
$$\begin{array}{l} \sqrt{x}\,.\,\sqrt{(x+1)}-\sqrt{(x-1)} \\ \sqrt{x}\,.\,\sqrt{(x+1)}-.\,\sqrt{(x-1)} \\ \hline x(x+1)-\sqrt{x}\,.\,\sqrt{(x^2-1)} \\ \qquad -\sqrt{x}\,.\,\sqrt{(x^2-1)}+(x-1) \\ \hline x^2+2x-1-2\sqrt{(x^3-x)} \end{array}$$

20.
$$\begin{array}{l} \sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\ \sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\ \hline x+1+\sqrt{x}.\sqrt{(x^2-1)} \\ \qquad +\sqrt{x}.\sqrt{(x^2-1)}+x(x-1) \\ \hline x^2+1+2\sqrt{(x^3-x)} \end{array}$$

CXV.

1. $c-d=(\sqrt{c}+\sqrt{d})(\sqrt{c}-\sqrt{d})$.
2. $c^2-d=(c+\sqrt{d})(c-\sqrt{d})$.
3. $c-d^2=(\sqrt{c}+d)(\sqrt{c}-d)$.
4. $1-y=(1+\sqrt{y})(1-\sqrt{y})$.
5. $1-3x^2=(1+\sqrt{3}.x)(1-\sqrt{3}.x)$.
6. $5m^2-1=(\sqrt{5}.m+1)(\sqrt{5}.m-1)$.
7. $4a^2-3x=\{2a+\sqrt{(3x)}\}\{2a-\sqrt{(3x)}\}$.
8. $9-8n=\{3+2\sqrt{(2n)}\}\{3-2\sqrt{(2n)}\}$.
9. $11n^2-16=(\sqrt{11}.n+4)(\sqrt{11}.n-4)$.
10. $p^2-4r=(p+2\sqrt{r})(p-2\sqrt{r})$.
11. $p-3q^2=(\sqrt{p}+\sqrt{3}.q)(\sqrt{p}-\sqrt{3}.q)$.

12. $a^{2m} - b^n = (a^m + b^{\frac{n}{2}})(a^m - b^{\frac{n}{2}})$.

13. $\dfrac{1}{a - \sqrt{b}} \times \dfrac{a + \sqrt{b}}{a + \sqrt{b}} = \dfrac{a + \sqrt{b}}{a^2 - b}$.

14. $\dfrac{\sqrt{a}}{\sqrt{a} - \sqrt{b}} \times \dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = \dfrac{a + \sqrt{(ab)}}{a - b}$.

15. $\dfrac{4 + 3\sqrt{2}}{3 - 2\sqrt{2}} \times \dfrac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} = \dfrac{12 + 17\sqrt{2} + 12}{9 - 8} = 24 + 17\sqrt{2}$.

16. $\dfrac{2}{2 - \sqrt{2}} \times \dfrac{2 + \sqrt{2}}{2 + \sqrt{2}} = \dfrac{4 + 2\sqrt{2}}{4 - 2} = \dfrac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2}$.

17. $\dfrac{\sqrt{3}}{2 - \sqrt{3}} \times \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}} = \dfrac{2\sqrt{3} + 3}{4 - 3} = 3 + 2\sqrt{3}$.

18. $\dfrac{2 - \sqrt{2}}{2 + \sqrt{2}} \times \dfrac{2 - \sqrt{2}}{2 - \sqrt{2}} = \dfrac{4 - 4\sqrt{2} + 2}{4 - 2} = \dfrac{6 - 4\sqrt{2}}{2} = 3 - 2\sqrt{2}$.

19. $\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}} \times \dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} \div \sqrt{x}} = \dfrac{a + 2\sqrt{(ax)} + x}{a - x}$.

20. $\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}} \times \dfrac{1 + \sqrt{x}}{1 + \sqrt{x}} = \dfrac{1 + 2\sqrt{x} + x}{1 - x}$.

21. $\dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} \times \dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} + \sqrt{(a-x)}}$

$= \dfrac{a + x + 2\sqrt{(a^2 - x^2)} + a - x}{a + x - (a - x)} = \dfrac{2a + 2\sqrt{(a^2 - x^2)}}{2x} = \dfrac{a + \sqrt{(a^2 - x^2)}}{x}$.

22. $\dfrac{\sqrt{(m^2+1)} - \sqrt{(m^2-1)}}{\sqrt{(m^2+1)} + \sqrt{(m^2-1)}} \times \dfrac{\sqrt{(m^2+1)} - \sqrt{(m^2-1)}}{\sqrt{(m^2+1)} - \sqrt{(m^2-1)}}$

$= \dfrac{m^2 + 1 - 2\sqrt{(m^4 - 1)} + m^2 - 1}{m^2 + 1 - (m^2 - 1)} = \dfrac{2m^2 - 2\sqrt{(m^4 - 1)}}{2}$

$= m^2 - \sqrt{(m^4 - 1)}$.

23. $\dfrac{a + \sqrt{(a^2 - 1)}}{a - \sqrt{(a^2 - 1)}} \times \dfrac{a + \sqrt{(a^2 - 1)}}{a + \sqrt{(a^2 - 1)}} = \dfrac{a^2 + 2a\sqrt{(a^2 - 1)} + a^2 - 1}{a^2 - (a^2 - 1)}$

$= 2a^2 - 1 + 2a\sqrt{(a^2 - 1)}$.

24. $$\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}\times\frac{a+\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}=\frac{a^2+2a\sqrt{(a^2-x^2)}+a^2-x^2}{a^2-(a^2-x^2)}$$
$$=\frac{2a^2-x^2+2a\sqrt{(a^2-x^2)}}{x^2}.$$

CXVI.

1. $$\begin{array}{l} 4+\sqrt{(-3)} \\ 4-\sqrt{(-3)} \\ \hline 16+4\sqrt{(-3)} \\ \quad -4\sqrt{(-3)}-(-3) \\ \hline 19 \end{array}$$

2. $$\begin{array}{l} \sqrt{3}-2\sqrt{(-2)} \\ \sqrt{3}+2\sqrt{(-2)} \\ \hline 3-2\sqrt{(-6)} \\ \quad +2\sqrt{(-6)}-4\times(-2) \\ \hline 11 \end{array}$$

3. $$\begin{array}{l} 4\sqrt{(-2)}-2\sqrt{2} \\ \frac{1}{2}\sqrt{(-2)}-3\sqrt{2} \\ \hline 2\times(-2)-\sqrt{(-4)} \\ \quad -12\sqrt{(-4)}+6\times 2 \\ \hline 8-13\sqrt{(-4)}=8-26\sqrt{(-1)} \end{array}$$

4. $$\begin{array}{l} \sqrt{(-2)}+\sqrt{(-3)}+\sqrt{(-4)} \\ \sqrt{(-2)}-\sqrt{(-3)}-\sqrt{(-4)} \\ \hline -2-\sqrt{6}-2\sqrt{2} \\ \quad +\sqrt{6}-(-3)+2\sqrt{3} \\ \quad\quad +2\sqrt{2}+2\sqrt{3}-(-4) \\ \hline -2+3+4\sqrt{3}+4=5+4\sqrt{3} \end{array}$$

5. $$\begin{array}{l} 3\sqrt{(-a)}+\sqrt{(-b)} \\ 4\sqrt{(-a)}-2\sqrt{(-b)} \\ \hline -12a-4\sqrt{(ab)} \\ \quad +6\sqrt{(ab)}+2b \\ \hline -12a+2\sqrt{(ab)}+2b \end{array}$$

6. $$\begin{array}{l} a+\sqrt{(-a)} \\ a-\sqrt{(-a)} \\ \hline a^2+a\sqrt{(-a)} \\ \quad -a\sqrt{(-a)}+a \\ \hline a^2+a \end{array}$$

7.
$$\begin{array}{l}
a\sqrt{(-a)}+b\sqrt{(-b)}\\
a\sqrt{(-a)}-b\sqrt{(-b)}\\
\hline
-a^3-ab\sqrt{(ab)}\\
\qquad +ab\sqrt{(ab)}+b^3\\
\hline
-a^3 \qquad\qquad +b^3
\end{array}$$

8.
$$\begin{array}{l}
a+\beta\sqrt{(-1)}\\
a-\beta\sqrt{(-1)}\\
\hline
a^2+a\beta\sqrt{(-1)}\\
\quad -a\beta\sqrt{(-1)}+\beta^2\\
\hline
a^2 \qquad\qquad +\beta^2
\end{array}$$

9.
$$\begin{array}{l}
1-\sqrt{(1-e^2)}\\
1+\sqrt{(1-e^2)}\\
\hline
1-\sqrt{(1-e^2)}\\
\quad +\sqrt{(1-e^2)}-(1-e^2)\\
\hline
\qquad\qquad e^2
\end{array}$$

10.
$$\begin{array}{l}
e^{p\sqrt{(-1)}}+e^{-p\sqrt{(-1)}}\\
e^{p\sqrt{(-1)}}-e^{-p\sqrt{(-1)}}\\
\hline
e^{2p\sqrt{(-1)}}+1\\
\qquad\qquad -1-e^{-2p\sqrt{(-1)}}\\
\hline
e^{2p\sqrt{(-1)}} \qquad -e^{-2p\sqrt{(-1)}}
\end{array}$$

CXVII.

1. $$\frac{3x+3\sqrt{(xy)}-3\sqrt{(xy)}+3y}{9\sqrt{(xy)}}=\frac{x+y}{3\sqrt{(xy)}}.$$

2. $$\{1+2\sqrt{(-1)}+(-1)\}+\{1-2\sqrt{(-1)}+(-1)\}$$
$$=2\sqrt{(-1)}-2\sqrt{(-1)}=0.$$

3. $$\frac{2\sqrt{(xy)}+2y+2x-2\sqrt{(xy)}}{4\sqrt{(xy)}}=\frac{x+y}{2\sqrt{(xy)}}.$$

4. $$\{1+2\sqrt{(-1)}-1\}-\{1-2\sqrt{(-1)}-1\}=4\sqrt{(-1)}=\sqrt{(-16)}.$$

5.
$$\begin{array}{l}
x^2+\sqrt{2}.ax+a^2)\,x^4+a^4\,(x^2-\sqrt{2}.ax+a^2\\
\qquad\qquad\qquad x^4+\sqrt{2}.ax^3+a^2x^2\\
\hline
\qquad\qquad\qquad -\sqrt{2}.ax^3-a^2x^2+a^4\\
\qquad\qquad\qquad -\sqrt{2}.ax^3-2a^2x^2-\sqrt{2}.a^3x\\
\hline
\qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4\\
\qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4\\
\hline
\end{array}$$

6. $$\begin{array}{l} m^2-\sqrt{2}.mn+n^2)\,m^4+n^4\,(m^2+\sqrt{2}.mn+n^2 \\ \qquad\qquad\quad \underline{m^4-\sqrt{2}.m^3n+m^2n^2} \\ \qquad\qquad\qquad\quad \sqrt{2}.m^3n-m^2n^2+n^4 \\ \qquad\qquad\qquad\quad \underline{\sqrt{2}.m^3n-2m^2n^2+\sqrt{2}.mn^3} \\ \qquad\qquad\qquad\qquad\qquad\quad m^2n^2-\sqrt{2}.mn^3+n^4 \\ \qquad\qquad\qquad\qquad\qquad\quad \underline{m^2n^2-\sqrt{2}.mn^3+n^4} \end{array}$$

7 $$\sqrt{x}.\sqrt{.}(x^2+2xy+y^2)+\sqrt{x}.\sqrt{(x^2-2xy+y^2)}$$
$$=\sqrt{x}.(x+y)+\sqrt{x}.(x-y)=2x\sqrt{x}.$$

8. $$\frac{a\sqrt{a}+a\sqrt{b}-b\sqrt{a}-b\sqrt{b}-(a\sqrt{a}+b\sqrt{a}-a\sqrt{b}-b\sqrt{b})}{a-b}$$
$$=\frac{2a\sqrt{b}-2b\sqrt{a}}{a-b}.$$

9. $$a^2.\frac{c}{b}-2a.\sqrt{\frac{c}{b}}.\sqrt{(cd)}+cd=\frac{a^2c}{b}-2ac\sqrt{\frac{d}{b}}+cd.$$

10. $$a^{2\sqrt{2}}-2a^{\sqrt{2}}.\frac{1}{a^{\sqrt{2}}}+\frac{1}{a^{2\sqrt{2}}}=a^{2\sqrt{2}}-2+\frac{1}{a^{2\sqrt{2}}}.$$

11. $$\frac{x^2+a^2+2\sqrt{(x^4-a^4)}+x^2-a^2+x^2+a^2-2\sqrt{(x^4-a^4)}+x^2-a^2}{(x^2+a^2)-(x^2-a^2)}$$
$$=\frac{4x^2}{2a^2}=\frac{2x^2}{a^2}.$$

12. $$\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1+x)}}\right\}\div\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x^2)}}\right\}=\frac{\sqrt{(1-x^2)}}{\sqrt{(1+x)}}=\sqrt{(1-x)}.$$

13. $$\frac{x-1}{x+1}\left\{\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}+\frac{(1+\sqrt{x})(1-\sqrt{x})}{\sqrt{x}(1+\sqrt{x})}\right\}$$
$$=\frac{x-1}{x+1}\left\{\sqrt{x}+1+\frac{1-\sqrt{x}}{\sqrt{x}}\right\}=\frac{x-1}{x+1}\left(\frac{x+\sqrt{x}+1-\sqrt{x}}{\sqrt{x}}\right)=\frac{x-1}{\sqrt{x}}.$$

14. $$\frac{x}{4}+3-2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{x}{2}-2\sqrt{\left(\frac{x^2}{16}-9\right)}.$$

15. $x+a-2\sqrt{(x^2-a^2)}+x-a=2x-2\sqrt{(x^2-a^2)}$.

16. $\sqrt[m]{(a^{2m-n+n}.b^{5m+1+m-1}.c^{3p+m-3p})}=\sqrt[m]{(a^{2m}.b^{6m}.c^{m})}=a^2b^6c$.

17. $\{-1-a\sqrt{(-1)}\}^2=1+2a\sqrt{(-1)}-a^2$

$$\{1+2a\sqrt{(-1)}-a^2\}^2=1-4a^2+a^4+4a\sqrt{(-1)}-2a^2-4a^3\sqrt{(-1)}$$

$$\begin{array}{l}
1-6a^2+a^4+4a\sqrt{(-1)}-4a^3\sqrt{(-1)} \\
-1-a\sqrt{(-1)} \\
\hline
-1+6a^2-a^4-4a\sqrt{(-1)}+4a^3\sqrt{(-1)} \\
\quad -a\sqrt{(-1)}+6a^3\sqrt{(-1)}-a^5\sqrt{(-1)}+4a^2-4a^4 \\
\hline
-1+10a^2-5a^4+(10a^3-a^5-5a)\sqrt{(-1)}
\end{array}$$

18. $3\sqrt[3]{3}-(-8)+4\sqrt[3]{3}=8+7\sqrt[3]{3}$.

19. $\dfrac{6c^2}{x-1}.\sqrt{\left(\dfrac{x(4x^2-8x+4)}{3c^3}\right)}=\dfrac{6c^2}{x-1}\cdot\dfrac{2x-2}{c}.\sqrt{\dfrac{x}{3c}}=\dfrac{12c\sqrt{x}}{\sqrt{(3c)}}=4\sqrt{(3cx)}$.

20. $\dfrac{x}{x-7}\{\sqrt[3]{[3p^2\{x^3-21x^2+147x-343\}]}\}=\dfrac{x}{x-7}\cdot\sqrt[3]{(3p^2)}.(x-7)$

$=x\sqrt[3]{(3p^2)}$.

21. $2(n-1)\sqrt[3]{\left[\dfrac{1}{2n}\cdot\left(-\dfrac{1}{n^3-3n^2+3n-1}\right)\right]}=2(n-1)\sqrt[3]{\left(\dfrac{1}{2n}\right)}\cdot\left(-\dfrac{1}{n-1}\right)$

$=-2\sqrt[3]{\dfrac{1}{2n}}=-\sqrt[3]{\dfrac{8}{2n}}=-\sqrt[3]{\dfrac{4n^2}{n^3}}=\dfrac{1}{n}\sqrt[3]{(-4n^2)}$.

22. $2(n-1).3\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{2(n-1)\times 5}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=6n\sqrt{7}-6\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{10n}{3}\sqrt{7}-\dfrac{10}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=9n\sqrt{7}-10\sqrt{7}$.

23. $\sqrt{(17^2-33)}\quad\sqrt[3]{(65^2-129)}=\sqrt{256}-\sqrt[3]{4096}=16-16=0$.

CXVIII.

Adopt the notation of Art. 316; then

1. $\left.\begin{aligned} x+y&=10 \\ xy&=21 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=100 \\ 4xy&=84 \end{aligned}\right\}$; $x^2-2xy+y^2=16$; $x-y=4$, etc.

2. $\left.\begin{aligned} x+y&=16 \\ xy&=55 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=256 \\ 4xy&=220 \end{aligned}\right\}$ $x^2-2xy+y^2=36$; $x-y=6$, etc.

3. $\left.\begin{aligned} x+y&=9 \\ xy&=14 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=81 \\ 4xy&=56 \end{aligned}\right\}$; $x^2-2xy+y^2=25$; $x-y=5$, etc.

4. $\left.\begin{aligned} x+y&=94 \\ xy&=2205 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=8836 \\ 4xy&=8820 \end{aligned}\right\}$; $x^2-2xy+y^2=16$;
$x-y=4$, etc.

5. $\left.\begin{aligned} x+y&=13 \\ xy&=30 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=169 \\ 4xy&=120 \end{aligned}\right\}$; $x^2-2xy+y^2=49$; $x-y=7$, etc.

6. $\left.\begin{aligned} x+y&=38 \\ xy&=360 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=1444 \\ 4xy&=1440 \end{aligned}\right\}$; $x^2-2xy+y^2=4$; $x-y=2$, etc.

7. $\left.\begin{aligned} x+y&=14 \\ xy&=24 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=196 \\ 4xy&=96 \end{aligned}\right\}$; $x^2-2xy+y^2=100$;
$x-y=10$, etc.

8. $\left.\begin{aligned} x+y&=103 \\ xy&=396 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=10609 \\ 4xy&=1584 \end{aligned}\right\}$; $x^2-2xy+y^2=9025$;
$x-y=95$, etc.

9. $\left.\begin{aligned} x+y&=75 \\ xy&=756 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=5625 \\ 4xy&=3024 \end{aligned}\right\}$; $x^2-2xy+y^2=2601$;
$x-y=51$, etc.

10. $\left.\begin{aligned} x+y&=87 \\ xy&=1512 \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=7569 \\ 4xy&=6048 \end{aligned}\right\}$; $x^2-2xy+y^2=1521$;
$x-y=39$, etc.

11. $\left.\begin{aligned} x+y&=\frac{7}{2} \\ xy&=\frac{10}{4} \end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=\frac{49}{4} \\ 4xy\quad&=\frac{40}{4} \end{aligned}\right\}$; $x^2-2xy+y^2=\frac{9}{4}$;

$x-y=\frac{3}{2}$, etc.

12. $\left.\begin{aligned} x+y&=\ 57 \\ xy&=540 \end{aligned}\right\}$ $\left.\begin{aligned} x^2+2xy+y^2&=3249 \\ 4xy\quad&=2160 \end{aligned}\right\}$; $x^2-2xy+y^2=1089$;

$x-y=33$, etc.

CXIX.

1. $x=7^2=49$. 2. $x=9^2=81$. 3. $x=5^2=25$.

4. $x=2^3=8$. 5. $x=3^3=27$. 6. $x=4^4=256$.

7. $x+9=36$, etc. 8. $x-7=49$, etc. 9. $x-15=64$, etc.

10. $x-9=144$, etc. 11. $4x-16=8$, etc. 12. $3\sqrt{x}=18$; $x=36$.

13. $\sqrt[3]{(2x+3)}=3$; $2x+3=27$; $2x=24$; $x=12$.

14. $c\sqrt{x}=a-b$; $c^2x=(a-b)^2$; $x=\frac{(a-b)^2}{c^2}$.

15. $\sqrt{(x^2-9)}=9-x$; $x^2-9=81-18x+x^2$; $18x=90$; $x=5$.

16. $x^2-11=x^2-2x+1$; $2x=12$; $x=6$.

17. $4x^2+5x-2=4x^2+4x+1$; $x=3$.

18. $9x^2-12x-51=9x^2-18x+9$; $6x=60$; $x=10$.

19. $x^2-ax+b=x^2+2ax+a^2$; $3ax=b-a^2$, etc.

20. $25x^2-3mx+n=m^2+10mx+25x^2$; $13mx=n-m^2$, etc.

CXX.

1. $\sqrt{(16+x)}=8-\sqrt{x}$; $16+x=64-16\sqrt{x}+x$; $16\sqrt{x}=48$; $\sqrt{x}=3$; $x=9$.

2. $x-16=64-16\sqrt{x}+x$; $16\sqrt{x}=80$; $\sqrt{x}=5$; $x=25$.

3. $\sqrt{(x+15)}=15-\sqrt{x}$; $x+15=225-30\sqrt{x}+x$; $30\sqrt{x}=210$; $\sqrt{x}=7$; $x=49$.

4. $x-21=x-2\sqrt{x}+1$; $2\sqrt{x}=22$; $\sqrt{x}=11$; $x=121$.

5. $x-1=x+4-6\sqrt{(x+4)}+9$; $6\sqrt{(x+4)}=14$; $\sqrt{(x+4)}=\frac{7}{3}$;
$x+4=\frac{49}{9}$; $x=\frac{13}{9}$.

6. $1+2\sqrt{(3x+1)}+3x+1=4x+4$; $2\sqrt{(3x+1)}=x+2$;
Squaring both sides, $4(3x+1)=x^2+4x+4$; $x^2=8x$, etc.

7. $1-2\sqrt{(1-3x)}+1-3x=4(1-x)$; $2\sqrt{(1-3x)}=x-2$;
$4(1-3x)=x^2-2x+4$; $x^2+8x=0$, etc.

8. $a-\sqrt{x}=\sqrt{(x-a)}$; $a^2-2a\sqrt{x}+x=x-a$; $2\sqrt{x}=1+a$;
$4x=(1+a)^2$, etc.

9. $\sqrt{(x-m)}=\frac{m-2\sqrt{x}}{2}$; $x-m=\frac{m^2-4m\sqrt{x}+4x}{4}$; $-4m=m^2-4m\sqrt{x}$;
$\sqrt{x}=\frac{m+4}{4}$; $x=\left(\frac{m+4}{4}\right)^2$.

10. $\sqrt{(x-4)}=3-\sqrt{(x-1)}$; $x-4=9-6\sqrt{(x-1)}+x-1$;
$6\sqrt{(x-1)}=12$; $\sqrt{(x-1)}=2$; $x-1=4$; $x=5$.

CXXI.

1. $\sqrt{(x^2-9x)}+x-9=36$; $\sqrt{(x^2-9x)}=45-x$;
$x^2-9x=2025-90x+x^2$; $81x=2025$; $x=25$.

2. $x+\sqrt{(x^2-21x)}=35$; $x^2-21x=1225-70x+x^2$; $49x=1225$ ·
$x=25$.

3. $x+7+\sqrt{(x^2+7x)}=28$; $\sqrt{(x^2+7x)}=21-x$;
$x^2+7x=441-42x+x^2$; $49x=441$; $x=9$.

4. $x - 15 + \sqrt{(x^2 - 15x)} = 105$; $\sqrt{(x^2 - 15x)} = 120 - x$;
$x^2 - 15x = 14400 - 240x + x^2$; $225x = 14400$; $x = 64$.

5. $\sqrt{(x^2 - 4x)} + x - 4 = 8$; $\sqrt{(x^2 - 4x)} = 12 - x$; $x^2 - 4x = 144 - 24x + x^2$;
$20x = 144$, etc.

6. $\sqrt{(3ax + x^2)} + 3a + x - 9a = 0$; $\sqrt{(3ax + x^2)} = 6a - x$;
$3ax + x^2 = 36a^2 - 12ax + x^2$; $15ax = 36a^2$, etc.

7. $b^2 - ax = b^2 - ax - ab + bx$; $bx = ab$; $x = a$.

8. $2 + \sqrt{x} - x = \dfrac{4 + \sqrt{x}}{2}$; $4 + 2\sqrt{x} - 2x = 4 + \sqrt{x}$; $\sqrt{x} = 2x$; $x = 4x^2$;
whence $x = 0$ or $\dfrac{1}{4}$.

9. $x + 28\sqrt{x} + 192 = x + 36\sqrt{x} + 128$; $8\sqrt{x} = 64$; $x = 64$.

10. $x - 6\sqrt{x} - 16 = x - 10\sqrt{x} + 24$; $4\sqrt{x} = 40$; $x = 100$.

CXXII.

1. $x - 3\sqrt{x} = 4$; $3\sqrt{x} = x - 4$; $9x = x^2 - 8x + 16$; $x^2 - 17x = -16$, etc.

2. $14\sqrt{x} = x + 45$; $196x = x^2 + 90x + 2205$; $x^2 - 106x = -2205$, etc.

3. $9(7 + 2x^3) = 25.(4x - 3)$; $63 + 18x^2 = 100x - 75$;
$18x^2 - 100x = -138$, etc.

4. $6x - 11 = 249 - 2x^2$; $2x^2 + 6x = 260$; $x^2 + 3x = 130$, etc.

5. $6 - x = 4 - 4\sqrt{(2x - 1)} + 2x - 1$; $4\sqrt{(2x - 1)} = 3x - 3$;
$16(2x - 1) = 9x^2 - 18x + 9$; $9x^2 - 50x = -25$, etc.

6. $x + 12 = 2\sqrt{(4 - 3x)}$; $x^2 + 24x + 144 = 4(4 - 3x)$; $x^2 + 36x = -128$.

7. $2x + 7 + 2\sqrt{(6x^2 - 15x - 126)} + 3x - 18 = 7x + 1$;
$\sqrt{(6x^2 - 15x - 126)} = x + 6$; $6x^2 - 15x - 126 = x^2 + 12x + 36$, etc.

8. $4(204-5x)=400-40\sqrt{(3x-68)}+3x-68$;
$40\sqrt{(3x-68)}=23x-484$;
$1600(3x-68)-529x^2-22264x+234256$;
$529x^2-27064x=-343056$;
$x^2-\frac{27064}{529}x+\frac{183115024}{(529)^2}=\frac{183115024-181476624}{(529)^2}=\frac{1638400}{(529)^2}$
$x-\frac{13532}{529}=\pm\frac{1280}{529}$, etc.

9. $(\sqrt{x}-4)(\sqrt{x}+4)=33$; $x-16=33$; $x=49$.

10. $(\sqrt{x}+11).(\sqrt{x}-11)=608$; $x-121=608$; $x=729$.

11. $\sqrt{(x^2+17x+60)}=12$; $x^2+17x+60=144$; $x^2+17x=-84$, etc.

12. $x+3+2\sqrt{(x^2+11x+24)}+x+8=25x$; $2\sqrt{(x^2+11x+24)}=23x-11$;
$4(x^2+11x+24)=529x^2-506x+121$; $525x^2-550x=-25$;
$x^2-\frac{22x}{21}=-\frac{1}{21}$, etc.

13. $\sqrt{(25+x)}=8-\sqrt{(25-x)}$; $25+x=64-16\sqrt{(25-x)}+25-x$;
$8\sqrt{(25-x)}=32-x$; $64(25-x)=1024-64x+x^2$; $x^2=576$;
$x=\pm 24$.

14. $x+4+2\sqrt{(2x^2+7x-4)}+2x-1=36$; $2\sqrt{(2x^2+7x-4)}=33-3x$,
$4(2x^2+7x-4)=1089-198x+9x^2$; $x^2-226x=-1105$, etc.

15. $\sqrt{(13x-1)}=5+\sqrt{(2x-1)}$; $13x-1=25+10\sqrt{(2x-1)}+2x-1$;
$11x-25=10\sqrt{(2x-1)}$; $121x^2-550x+625=100(2x-1)$;
$121x^2-750x=-725$; $x^2-\frac{750}{121}x+\frac{140625}{(121)^2}=\frac{140625-87725}{(121)^2}$, etc.

16. $7x+1-2\sqrt{(21x^2+10x+1)}+3x+1=4$; $\sqrt{(21x^2+10x+1)}=5x-1$;
$21x^2+10x+1=25x^2-10x+1$; $4x^2=20x$; $x=5$ or 0.

17. $\sqrt{(4+x)}=3-\sqrt{x}$; $4+x=9-6\sqrt{x}+x$; $\sqrt{x}=\frac{5}{6}$; $x=\frac{25}{36}$.

18. $x+\sqrt{(x^2+9975)}=525$; $\sqrt{(x^2+9975)}=525-x$;
$x^2+9975x=275625-1050x+x^2$; $11025x=275625$, etc.

19. $\frac{x}{4}+3+2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{2x}{3}$; $2\sqrt{\left(\frac{x^2}{16}-9\right)}=\frac{x}{6}$;

$\frac{x^2}{16}-9=\frac{x^2}{144}$; $\frac{8x^2}{144}=9$; $x^2=9\times 18$; $x=\pm 9\sqrt{2}$.

20. $x^2-1+6\sqrt{(x^2-1)}=16$; $x^2-1+6\sqrt{(x^2-1)}+9=25$;
$\sqrt{(x^2-1)}+3=\pm 5$; $\sqrt{(x^2-1)}=2$ or -8.
Hence $x^2-1=4$ or 64; $x^2=5$ or 65, etc.

21. $x^2-2ax+a^2+2ab+b^2=x^2+a^2+b^2-2ax+2bx-2ab$; $4ab=2bx$;
$x=2a$.

22. $x^2+2ax+a^2+2ab+b^2=b^2+a^2+x^2-2ab-2bx+2ax$; $4ab=-2bx$;
$x=-2a$.

23. $x+4-2\sqrt{(x^2+4x)}+x=x+\frac{3}{2}$; $x+\frac{5}{2}=2\sqrt{(x^2+4x)}$;

$x^2+5x+\frac{25}{4}=4x^2+16x$; $3x^2+11x=\frac{25}{4}$; $x^2+\frac{11x}{3}+\frac{121}{36}=\frac{196}{36}$;

$x+\frac{11}{6}=\pm\frac{14}{6}$; $x=\frac{1}{2}$ or $-\frac{25}{6}$.

24. $\sqrt{x}+1=x+\frac{5}{4}$; $\sqrt{x}=x+\frac{1}{4}$; $x-\sqrt{x}=-\frac{1}{4}$; $x-\sqrt{x}+\frac{1}{4}=0$;

$\sqrt{x}-\frac{1}{2}=0$; $x=\frac{1}{4}$.

25. $\sqrt{(4+x)}=\sqrt{3}+\sqrt{x}$; $4+x=3+2\sqrt{(3x)}+x$; $2\sqrt{(3x)}=1$;
$12x=1$, etc.

26. $\sqrt{(x+4)}=9-\sqrt{(x+5)}$; $x+4=81-18\sqrt{(x+5)}+x+5$;
$18\sqrt{(x+5)}=82$; $9\sqrt{(x+5)}=41$; $81(x+5)=1681$, etc.

27. $\sqrt{(x^2-4x)}+x-4=8$; $\sqrt{(x^2-4x)}=12-x$; $x^2-4x=144-24x+x^2$;
$20x=144$, etc.

28. $x^2-21=\sqrt{(x^2-9)}$; $x^4-42x^2+441=x^2-9$; $x^4-43x^2=-450$;

$x^4-43x^2+\frac{1849}{4}=\frac{49}{4}$; $x^2-\frac{43}{2}=\pm\frac{7}{2}$; $x^2=25$ or 18, etc.

29. $\sqrt{(50+x)}=2+\sqrt{(50-x)}$; $50+x=4+4\sqrt{(50-x)}+50-x$;
$x-2=2\sqrt{(50-x)}$; $x^2-4x+4=4(50-x)$; $x^2=196$, etc.

30. $\sqrt{(2x+4)}-1=\sqrt{\left(\frac{x}{2}+6\right)}$; $2x+4-2\sqrt{(2x+4)}+1=\frac{x}{2}+6$;

$\frac{3x}{2}-1=2\sqrt{(2x+4)}$; $9x^2-12x+4=16(2x+4)$; $9x^2-44x=60$;

$x^2-\frac{44}{9}x+\frac{484}{81}=\frac{1024}{81}$, etc.

31. $3+x+\sqrt{(3x+x^2)}=6$; $\sqrt{(3x+x^2)}=3-x$; $3x+x^2=9-6x+x^2$;
$9x=9$; $x=1$.

32. $\sqrt{(x-1)}+\sqrt{(x+1)}=1$; $\sqrt{(x-1)}=1-\sqrt{(x+1)}$;
$x-1=1-2\sqrt{(x+1)}+x+1$; $2\sqrt{(x+1)}=3$; $4(x+1)=9$, etc.

33. $3x+\sqrt{(4x-x^2)}=6x-2\sqrt{(4x-x^2)}$; $3\sqrt{(4x-x^2)}=3x$; $4x-x^2=x^2$;
$2x^2=4x$, etc.

34. $\sqrt{\{a-\sqrt{(ax+x^2)}\}}=\sqrt{a}-\sqrt{x}$; $a-\sqrt{(ax+x^2)}=a-2\sqrt{(ax)}+x$;
$\sqrt{(ax+x^2)}=2\sqrt{(ax)}-x$; $ax+x^2=4ax-4x\sqrt{(ax)}+x^2$;
$4x\sqrt{(ax)}=3ax$; whence $x=0$, or $4\sqrt{(ax)}=3a$; $16ax=9a^2$;
$16x=9a$, etc.

CXXIII.

1. $x-2=0$, or, $x-5=0$; $\therefore x=2$ or 5.

2. $x-3=0$, or, $x+7=0$; $\therefore x=3$ or -7.

3. $x+9=0$, or, $x+2=0$; $\therefore x=-9$ or -2.

4. $x-5a=0$, or, $x-6b=0$; $\therefore x=5a$ or $6b$.

5. $2x+7=0$, or, $3x-5=0$; $\therefore x=-\frac{7}{2}$ or $\frac{5}{3}$.

6. $19x-227=0$, or, $14x+83=0$; $\therefore x=\frac{227}{19}$ or $-\frac{83}{14}$.

7. $5x-4m=0$, or, $6x-11n=0$; $\therefore x=\frac{4m}{5}$ or $\frac{11n}{6}$.

8. $x^2+5ax+6a^2=0$, or, $x^2-7ax+12a^2=0$;
$(x+2a)(x+3a)=0$, or, $(x-4a)(x-3a)=0$;
$\therefore x+2a=0$, or, $x+3a=0$, or, $x-4a=0$, or, $x-3a=0$, etc.

9. $(x+2)(x-2)(x-a)(x-a)=0$; $\therefore x=\pm 2$ or a.

10. $x.x.(x-5)=0$, $\therefore x=0$ or 5.

11. $acx-2a+b=0$, or, $bcx+3a-b=0$;
$\therefore acx=2a-b$, or, $bcx=b-3a$, etc

12. $cx-d=0$, or, $cx-e=0$, etc.

CXXIV.

1. Let m be one of the roots of the first equation, and $\frac{1}{m}$ one of the roots of the second equation.
Then $am^2+bm+c=0$, and $c'm^2+b'm+a'=0$.
Multiply the first by a' and the second by c; then
$aa'm^2+a'bm+a'c=0$, and $cc'm^2+b'cm+a'c=0$.
Subtracting $(aa'-cc')m^2+(a'b-b'c)m=0$.
Whence $(aa'-cc')m=-(a'b-b'c)$ . . . . (1).
Again, multiply the first by c' and the second by a; then
$ac'm^2+bc'm+cc'=0$, and $ac'm^2+ab'm+aa'=0$.
Subtracting $(bc'-ab')m=cc'-aa'$;
or $aa'-cc'=-(ab'-bc')m$ . . . . (2).
Multiplying (1) and (2) together, we get
$(aa'-cc')^2=(ab'-bc')(a'b-b'c)$.

2. $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$;
$\therefore \alpha^2+2\alpha\beta+\beta^2=\frac{b^2}{a^2}$, and $2\alpha\beta=\frac{2c}{a}$
$\therefore \alpha^2+\beta^2=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$.

3. $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

$$\therefore ac\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=ac\left(x^2-\frac{\alpha^2+\beta^2}{\alpha\beta}x+1\right)$$
$$=ac\left(x^2-\frac{b^2-2ac}{ac}x+1\right)$$
$$=acx^2+(2ac-b^2)x+ac.$$

4. The roots are $\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$, and that these may be *equal*, we must have $b^2-4ac=0$, or $c=\dfrac{b^2}{4a}$

Putting this for c in the expression ax^2+bx+c,

$$ax^2+bx+\frac{b^2}{4a}=\frac{4a^2x^2+4abx+b^2}{4a}=\left(\frac{2ax+b}{2\sqrt{a}}\right)^2.$$

5. $\alpha+\beta=1+a$, and $\alpha\beta=\dfrac{1+a+a^2}{2}$.

$\alpha^2+2\alpha\beta+\beta^2=1+2a+a^2$, and $2\alpha\beta=1+a+a^2$

$\therefore \alpha^2+\beta^2=a.$

CXXV.

1. $(x-5)(x-6)=0$, or, $x^2-11x+30=0$.

2. $(x-4)(x+5)=0$, or, $x^2+x-20=0$.

3. $(x+2)(x+7)=0$, or, $x^2+9x+14=0$.

4. $\left(x-\dfrac{1}{2}\right)\left(x-\dfrac{2}{3}\right)=0$, or, $(2x-1)(3x-2)=0$, or, $6x^2-7x+2=0$.

5. $(x-7)\left(x+\dfrac{5}{9}\right)=0$, or, $(x-7)(9x+5)=0$, or, $9x^2-58x-35=0$.

6. $(x-\sqrt{3})(x+\sqrt{3})=0$, or, $x^2-3=0$.

7. $(x-m-n)(x-m+n)=0$, or, $x^2-2mx+m^2-n^2=0$.

8. $\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)=0$, or, $x^2-\frac{\alpha+\beta}{\alpha\beta}x+\frac{1}{\alpha\beta}=0$.

9. $\left(x+\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$, or, $x^2+\frac{\alpha^2-\beta^2}{\alpha\beta}x-1=0$.

CXXVI.

1. One root is found by trial to be 2; then
$$x^3-11x^2+36x-36=(x-2)(x^2-9x+18)=(x-2)(x-3)(x-6).$$

2. One root is found by trial to be 1; then
$$x^3-7x^2+14x-8=(x-1)(x^2-6x+8)=(x-1)(x-2)(x-4).$$

3. One root is found by trial to be -1; then
$$x^3-5x^2-46x-40=(x+1)(x^2-6x-40)=(x+1)(x-10)(x+4).$$

4. One root is found by trial to be -1; then
$$4x^3+6x^2+x-1=(x+1)(4x^2+2x-1)=(x+1)4.\left(x^2+\frac{x}{2}-\frac{1}{4}\right)$$
and since the roots of $x^2+\frac{x}{2}-\frac{1}{4}=0$ are $\frac{\sqrt{5}-1}{4}$ and $\frac{-\sqrt{5}-1}{4}$,
$$x^2+\frac{x}{2}-\frac{1}{4}=\left(x+\frac{1-\sqrt{5}}{4}\right).\left(x+\frac{1+\sqrt{5}}{4}\right),$$ and therefore
$$4x^3+6x^2+x-1=4(x+1)\left(x+\frac{1-\sqrt{5}}{4}\right)\left(x+\frac{1+\sqrt{5}}{4}\right).$$

5. One root is found by trial to be -1; then
$$6x^3+11x^2-9x-14=(x+1)(6x^2+5x-14)=(x+1)(x+2)(6x-7).$$

6. If we put $x=-y-z$, the expression becomes zero, which shows that $x+y+z$ is one of its factors; the other is found by division to be $x^2+y^2+z^2-xy-xz-yz$.

7. If we put $a=b+c$, the expression becomes zero, which shows that $a-b-c$ is one of its factors; the other is found by division to be $a^2+b^2+c^2+ab+ac-bc$.

8. One root is found by trial to be 1; then

$$3x^3-x^2-23x+21=(x-1)(3x^2+2x-21)=(x-1)(3x-7)(x+3).$$

9. One root is found by trial to be 1; then

$$2x^3-5x^2-17x+20=(x-1)(2x^2-3x-20)=(x-1)(x-4)(2x+5).$$

10. One root is found by trial to be -1; then

$$15x^3+41x^2+5x-21=(x+1)(15x^2+26x-21)$$
$$=(x+1)(3x+7)(5x-3).$$

CXXVII.

1. $x^4-12x^2=13$; $x^4-12x^2+36=49$; $x^2-6=\pm 7$, etc.

2. $x^6+14x^3=-24$; $x^6+14x^3+49=25$; $x^3+7=\pm 5$, etc.

3. $x^8+22x^4=-21$; $x^8+22x^4+121=100$; $x^4+11=\pm 10$, etc

4. $x^{2m}+3x^m+\frac{9}{4}=\frac{25}{4}$; $x^m+\frac{3}{2}=\pm\frac{5}{2}$, etc.

5. $x^{4n}-\frac{5}{3}x^{2n}+\frac{25}{36}=\frac{100}{36}$; $x^{2n}-\frac{5}{6}=\pm\frac{10}{6}$, etc.

6. $x-\frac{9}{2}x^{\frac{1}{2}}+\frac{81}{16}=\frac{121}{16}$; $x^{\frac{1}{2}}-\frac{9}{4}=\pm\frac{11}{4}$, etc.

7. $x^{-2}+3x^{-1}+\frac{9}{4}=\frac{97}{36}$; $x^{-1}+\frac{3}{2}=\pm\frac{\sqrt{97}}{6}$; $x^{-1}=\frac{-9\pm\sqrt{97}}{6}$

$$\therefore x=\frac{6}{-9\pm\sqrt{97}}.$$

8. $x^{-2n}-x^{-n}+\frac{1}{4}=\frac{81}{4}$; $x^{-n}-\frac{1}{2}=\pm\frac{9}{2}$; $x^{-n}=5$ or -4,

$$\therefore \frac{1}{x^n}=\frac{1}{5} \text{ or } -\frac{1}{4};\ x=\left(\frac{1}{5}\right)^{\frac{1}{n}} \text{ or } \left(-\frac{1}{4}\right)^{\frac{1}{n}}.$$

9. $x^2-2x+5+6(x^2-2x+5)^{\frac{1}{2}}=16$;

$(x^2-2x+5)+6(x^2-2x+5)^{\frac{1}{2}}+9=25$; $(x^2-2x+5)^{\frac{1}{2}}+3=\pm 5$;

$(x^2-2x+5)^{\frac{1}{2}}=2$ or -8; $x^2-2x+5=4$ or 64, etc.

10. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;
$2x^2-5x+6+10\sqrt{(2x^2-5x+6)}=39$;
$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}+25=64$;
$\sqrt{(2x^2-5x+6)}+5=\pm 8$; $\sqrt{(2x^2-5x+6)}=3$ or -13;
$2x^2-5x+6=9$ or 169, etc.

11. $3x^2-6\sqrt{(3x^2-2ax+4)}+12=2ax+a^2+2a$;
$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2a-8$;
$(3x^2-2ax+4)-6\sqrt{(3x^2-2ax+4)}+9=a^2+2a+1$;
$\sqrt{(3x^2-2ax+4)}-3=\pm(a+1)$;
$3x^2-2ax+4=(4+a)^2$ or $(2-a)^2$, etc.

12. $x^2-ax+a^2-2\sqrt{(x^2-ax+a^2)}=-2a+a^2$;
$(x^2-ax+a^2)-2\sqrt{(x^2-ax+a^2)}+1=a^2-2a+1$;
$\sqrt{(x^2-ax+a^2)}-1=\pm(a-1)$; $x^2-ax+a^2=a^2$ or $(2-a)^2$, etc.

CXXVIII.

1. $\frac{2}{3}, \frac{6}{7}, \frac{7}{9}; \frac{42}{63}, \frac{54}{63}, \frac{49}{63}$, etc.

2. $\frac{x+3y}{x+2y}, \frac{x+2y}{x+y}; \frac{x^2+4xy+3y^2}{(x+2y)(x+y)}, \frac{x^2+4xy+4y^2}{(x+2y)(x+y)}$, etc.

3. $\frac{x-5y}{x-4y}, \frac{x-3y}{x-2y}; \frac{x^2-7xy+10y^2}{(x-4y)(x-2y)}, \frac{x^2-7xy+12y^2}{(x-4y)(x-2y)}$, etc.

4. Let x be the number.

Then $\frac{a+x}{b+x}=\frac{c}{d}$; $ad+dx=bc+cx$;

$(c-d)x=ad-bc$; $x=\frac{ad-bc}{c-d}$.

5. Let $x:y$ be the ratio. Then

$$\left.\begin{array}{l} x^2+y^2=181 \\ xy=90 \end{array}\right\}, \left.\begin{array}{l} x^2+y^2+2xy=361 \\ x^2+y^2-2xy=1 \end{array}\right\}, \left.\begin{array}{l} x+y=19 \\ x-y=\pm 1 \end{array}\right\}, \begin{array}{l} x=10 \text{ or } 9 \\ y=9 \text{ or } 10 \end{array}$$

CXXIX.

1. $x:y=6:9=2:3$.

2. $x:y=b:a$.

3. $(a-c)x=(b+d)y$; $x:y=b+d:a-c$.

4. $x^2+2xy+y^2=6y^2$; $x+y=\pm\sqrt{6}.y$;

$x=(\pm\sqrt{6}-1)y$; $x:y=\pm\sqrt{6}-1:1$.

5. $x^2-12xy+36y^2=49y^2$; $x-6y=\pm 7y$;

$x=13y$ or $-y$; $x:y=13:1$, or, $-1:1$.

6. $x^2+mxy+\frac{m^2y^2}{4}=\frac{(m^2+4n^2)y^2}{4}$; $x+\frac{my}{2}=\frac{\pm\sqrt{(m^2+4n^2)}y}{2}$;

$x=\frac{-m\pm\sqrt{(m^2+4n^2)}}{2}y$, etc.

7. Let $3x$ and $4x$ be the numbers. Then

$\frac{3x+4x}{9x^2+16x^2}=\frac{7}{50}$; $7\times 25x^2=50\times 7x$; $x=2$.

Hence the numbers are 6 and 8.

8. Let $6x$ and $7x$ be the numbers. Then

$\frac{6x+12}{7x+12}=\frac{12}{13}$; $78x+156=84x+144$; $6x=12$; $x=2$, etc.

9. Let $7x$ and $13x$ be the numbers. Then $20x=100$; $x=5$, etc.

10. Let x and y be the numbers. Then $x^2-y^2=48$, and $\frac{x+y}{x-y}=\frac{12}{1}$.

Hence $x+y=12x-12y$, or, $13y=11x$.

Then $x^2-\frac{121x^2}{169}=48$; $48x^2=169\times 48$; $x=13$, etc.

11. Let x be the value of a gold coin, and y the value of a silver one.
Then $5x+4y=3x+12y$; $2x=8y$; $x:y=4:1$.

12. Let x be the value of a gold coin, and y the value of a silver one.
Then $8x+9y=6x+19y$; $2x=10y$; $y:x=1:5$.

CXXX.

1. $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

2. $\frac{3}{7} \times \frac{14}{9} \times \frac{4}{3} = \frac{8}{9}$.

3. $\frac{x^2 - y^2}{x^3 + y^3} \times \frac{x^2 - xy + y^2}{x + y} = \frac{x - y}{x + y}$.

4. $\frac{a^2 - b^2 + 2bc - c^2}{a^2 - b^2 - 2bc - c^2} \times \frac{a + b + c}{a + b - c} = \frac{(a + b - c)(a - b + c)(a + b + c)}{(a + b + c)(a - b - c)(a + b - c)}$, etc

5. $\frac{m^3 + n^3}{m^3 - n^3} \times \frac{m - n}{m + n} = \frac{m^2 - mn + n^2}{m^2 + mn + n^2}$.

6. $\frac{(x + 2)(x + 3)}{(y - 3)(y - 4)} \times \frac{y(y - 3)}{x(x + 3)} = \frac{(x + 2)y}{(y - 4)x}$.

CXXXI.

1. Let $\frac{a}{b} = \lambda$, then $\frac{c}{d} = \lambda$. Then $a = \lambda b$ and $c = \lambda d$.

Then $\frac{a + b}{a} = \frac{\lambda b + b}{\lambda b} = \frac{\lambda + 1}{\lambda}$,

and $\frac{c + d}{c} = \frac{\lambda d + d}{\lambda d} = \frac{\lambda + 1}{\lambda}$.

2. Let $\frac{a}{b} = \lambda$, then $\frac{c}{d} = \lambda$. Then $a = \lambda b$, $c = \lambda d$.

Then $\frac{a^2 - b^2}{b^2} = \frac{\lambda^2 b^2 - b^2}{b^2} = \lambda^2 - 1$,

and $\frac{c^2 - d^2}{d^2} = \frac{\lambda^2 d^2 - d^2}{d^2} = \lambda^2 - 1$.

3. Let $\frac{a_1}{b_1} = \lambda$, then $\frac{a_2}{b_2} = \lambda$. Then $a_1 = \lambda b_1$, and $a_2 = \lambda b_2$.

Then $\frac{m_1 a_1 + m_2 a_2}{m_1 b_1 + m_2 b_2} = \frac{m_1 \lambda b_1 + m_2 \lambda b_2}{m_1 b_1 + m_2 b_2} = \lambda = \frac{a_1}{b_1}$.

4. Let $a=b\lambda$ and $c=d\lambda$. Then

$$\frac{3a^2+ab+2b^2}{3a^2-2b^2}=\frac{3b^2\lambda^2+b^2\lambda+2b^2}{3b^2\lambda^2-2b^2}=\frac{3\lambda^2+\lambda+2}{3\lambda^2-2}$$

$$\frac{3c^2+cd+2d^2}{3c^2-2d^2}=\frac{3d^2\lambda^2+d^2\lambda+2d^2}{3d^2\lambda^2-2d^2}=\frac{3\lambda^2+\lambda+2}{3\lambda^2-2}.$$

5. Let $a=b\lambda$ and $c=d\lambda$. Then

$$\frac{a^2+3ab+b^2}{c^2+3cd+d^2}=\frac{b^2\lambda^2+3b^2\lambda+b^2}{d^2\lambda^2+3d^2\lambda+d^2}=\frac{b^2(\lambda^2+3\lambda+1)}{d^2(\lambda^2+3\lambda+1)}=\frac{b^2}{d^2}$$

$$\frac{2ab+3b^2}{2cd+3d^2}=\frac{2b^2\lambda+3b^2}{2d^2\lambda+3d^2}=\frac{b^2(2\lambda+3)}{d^2(2\lambda+3)}=\frac{b^2}{d^2}.$$

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}=\lambda$ and $\frac{e}{f}=\lambda$.

Then $a=b\lambda$, $c=d\lambda$, $e=f\lambda$.

Then $\frac{mc-ne}{md-nf}=\frac{md\lambda-nf\lambda}{md-nf}=\lambda=\frac{a}{b}$.

7. $$\frac{a-\frac{ma}{n}}{b-\frac{mb}{n}}=\frac{a\left(1-\frac{m}{n}\right)}{b\left(1-\frac{m}{n}\right)}=\frac{a}{b}.$$

8. Let $a=b\lambda$, $c=d\lambda$, $e=f\lambda$.

Then $\frac{ac}{bd}=\frac{bd\lambda^2}{bd}=\lambda^2$

and $\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}=\frac{lb^2\lambda^2+md^2\lambda^2+nf^2\lambda^2}{lb^2+md^2+nf^2}=\lambda^2$.

9. Let $a_1=b_1\lambda$; $a_2=b_2\lambda$; $a_3=b_3\lambda$.

Then $\frac{a_1^2+a_2^2+a_3^2}{b_1^2+b_2^2+b_3^2}=\frac{b_1^2\lambda^2+b_2^2\lambda^2+b_3^2\lambda^2}{b_1^2+b_2^2+b_3^2}=\lambda^2=\frac{a_1^2}{b_1^2}$.

10. Let $a_1=b_1\lambda$; $a_2=b_2\lambda$; $a_3=b_3\lambda$.

Then $\frac{a_1a_2+a_2a_3+a_3a_1}{b_1b_2+b_2b_3+b_3b_1}=\frac{b_1b_2\lambda^2+b_2b_3\lambda^2+b_3b_1\lambda^2}{b_1b_2+b_2b_3+b_3b_1}=\lambda^2=\frac{a_1^2}{b_1^2}$.

11. $(a^2-ab+b^2)(c^2+cd+d^2)=(a^2+ab+b^2)(c^2-cd+d^2)$;
$(a^2+b^2)(c^2+cd+d^2-c^2+cd-d^2)=ab(c^2-cd+d^2+c^2+cd+d^2)$;
$(a^2+b^2)(2cd)=ab(2c^2+2d^2)$; $a^2cd+b^2cd=abc^2+abd^2$;
$a^2cd-abd^2=abc^2-b^2cd$; $ad(ac-bd)=bc(ac-bd)$;
then *either* $ad=bc$; and therefore $a:b=c:d$,
or $ac-bd=0$; and therefore $ac=bd$, or, $a:b=d:c$.

12. $(a^2+b^2)(c^2-d^2)=(a^2-b^2)(c^2+d^2)$;
$a^2c^2+b^2c^2-a^2d^2-b^2d^2=a^2c^2+a^2d^2-b^2c^2-b^2d^2$; $2b^2c^2=2a^2d^2$;
$bc=ad$; and $\therefore a:b=c:d$.

13. Let $\frac{a}{b}=\lambda$ and $\frac{c}{d}=\lambda$; then $a=b\lambda$ and $c=d\lambda$.

Then $\frac{(a+c)(a^2+c^2)}{(a-c)(a^2-c^2)}=\frac{(b\lambda+d\lambda)(b^2\lambda^2+d^2\lambda^2)}{(b\lambda-d\lambda)(b^2\lambda^2-d^2\lambda^2)}=\frac{(b+d)(b^2+d^2)}{(b-d)(b^2-d^2)}$.

14. Let $a_1=b_1\lambda$ and $a_2=b_2\lambda$.

Then $\frac{\sqrt{(a_1^2+a_2^2)}}{\sqrt{(b_1^2+b_2^2)}}=\sqrt{\frac{b_1^2\lambda^2+b_2^2\lambda^2}{b_1^2+b_2^2}}=\sqrt{(\lambda^2)}=\lambda=\frac{a_1}{b_1}$.

CXXXII.

1. $(a-b)c=(b-c)b$; $ac-bc=b^2-bc$; $ac=b^2$; $a:b=b:c$.

2. Let $a=\lambda b$ and $c=\lambda d$.

Then $\frac{(a^2+b^2)(a+b)}{a^3}=\frac{(\lambda^2b^2+b^2)(\lambda b+b)}{\lambda^3b^3}=\frac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$,

and $\frac{(c^2+d^2)(c+d)}{c^3}=\frac{(\lambda^2d^2+d^2)(\lambda d+d)}{\lambda^3d^3}=\frac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$.

Also, $\frac{\sqrt[4]{(ma^4+nc^4)}}{\sqrt[4]{(mb^4+nd^4)}}=\sqrt[4]{\frac{mb^4\lambda^4+nd^4\lambda^4}{mb^4+nd^4}}=\sqrt[4]{\lambda^4}=\lambda=\frac{a}{b}$.

3. Let $a=b\lambda$ and $c=d\lambda$.

Then $\frac{ma-nb}{ma+nb}=\frac{mb\lambda-nb}{mb\lambda+nb}=\frac{m\lambda-n}{m\lambda+n}$,

and $\frac{mc-nd}{mc+nd}=\frac{md\lambda-nd}{md\lambda+nd}=\frac{m\lambda-n}{m\lambda+n}$.

4. $(5a+3b)(7b+3c)=(7a+3b)(5b+3c)$;
$35ab+15ac+21b^2+9bc=35ab+21ac+15b^2+9bc$; $6b^2=6ac$;
$b^2=ac$; $a:b=b:c$.

5. Let $a:b=c:d$, and suppose a to be the greatest of the four.
Then $\because \frac{a}{b}=\frac{c}{d}$, and a is greater than b, c is greater than d,
and $\because \frac{a}{c}=\frac{b}{d}$, and a is greater than c, b is greater than d.
Again, if $a+b:m+n=m-n:a-b$,
and $a+b$ is greater than $m+n$
$m-n$ is greater than $a-b$
$\therefore a+b+m-n$ is greater than $m+n+a-b$
$\therefore 2b$ is greater than $2n$; and $\therefore b$ is greater than n.

6. $(x-1)(x+2)=(x-2)(2x+1)$; $x^2+x-2=2x^2-3x-2$; $x^2=4x$;
$x=4$ or 0.

7. $\frac{a}{b}+1=\frac{c}{d}+1$; $\frac{a}{b}=\frac{c}{d}$.

8. Suppose the bicycle went $5x$ and the tricycle $4x$ yards per minute.
Then the tricycle had a start of $2x$ yards out of 1760 yards.
Also the bicycle went 880 yards while the tricycle went
$1760-2x-176$. $\therefore 1760:1760-2x-176=5:4$;
$1760\times 4=(1584-2x)\times 5$; $10x=7920-7040$, etc.

9. $\frac{a}{b}=\frac{c}{d}$; $\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}$; $\therefore \frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}$;
$\therefore \frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}$; and since b^2 is greater than d^2,
$\therefore a^2-b^2$ is greater than c^2-d^2
$\therefore a^2+d^2$ is greater than b^2+c^2.

10. $(10a+b)(12c+d)=(10c+d)(12a+b)$;
$120ac+12bc+10ad+bd=120ac+12ad+10bc+bd$; $2bc=2ad$
$bc=ad$; $a:b=c:d$.

11. $2x=3y$, and $xy=600$.

Hence $x \times \frac{2x}{3}=600$; $x^2=900$; $x=30$, etc.

12. Let $\frac{a}{b}=\lambda$; then $\frac{b}{c}=\lambda$.

Then (1.) $\frac{a}{a+b}=\frac{\lambda b}{\lambda b+b}=\frac{\lambda}{\lambda+1}$

$$\frac{a-b}{a-c}=\frac{\lambda b-b}{\lambda b-c}=\frac{\lambda^2 c-\lambda c}{\lambda^2 c-c}=\frac{\lambda^2-\lambda}{\lambda^2-1}=\frac{\lambda}{\lambda+1}.$$

(2.) $(a^2+b^2)(b^2+c^2)=(\lambda^2 b^2+b^2)(\lambda^2 c^2+c^2)$

$$=\lambda^2 c^2(\lambda^2+1)(\lambda^2 c^2+c^2)=\lambda^2 c^4(\lambda^2+1)^2$$

$$(ab+bc)^2=(\lambda b^2+\lambda c^2)^2=(\lambda^3 c^2+\lambda c^2)^2=\lambda^2 c^4(\lambda^2+1)^2.$$

13. Let $a=b\lambda$ and $c=d\lambda$.

Then $\frac{a+b}{b}=\frac{b\lambda+b}{b}=\lambda+1$;

and $\frac{c+d}{d}=\frac{d\lambda+d}{d}=\lambda+1$.

Hence $\frac{ab-bc-dx+(bc+dx)}{bc+dx}=\frac{a-b-c+(b+c)}{b+c}$;

or, $\frac{ab}{bc+dx}=\frac{a}{b+c}$; $\frac{b}{bc+dx}=\frac{1}{b+c}$; $b^2+bc=bc+dx$, etc.

14. Let $a=\lambda b$, and $b=\lambda c$.

Then $\frac{a+mb}{a-mb}=\frac{\lambda b+mb}{\lambda b-mb}=\frac{\lambda+m}{\lambda-m}$;

and $\frac{b+mc}{b-mc}=\frac{\lambda c+mc}{\lambda c-mc}=\frac{\lambda+m}{\lambda-m}$.

15. $a=\frac{5b}{4}$; and therefore $\frac{a^2-b^2}{a^2+b^2}=\frac{\frac{25b^2}{16}-b^2}{\frac{25b^2}{16}+b^2}=\frac{9}{41}$.

16. Let the sides be $2\frac{1}{2}x$, $3\frac{3}{4}x$, and $4x$ yards.

Then $2\frac{1}{2}x+3\frac{3}{4}x+4x=205$; $\frac{41}{4}x=205$; $x=20$, etc.

17. Let the sides be $3x$, $4x$ and $5x$ yards.

Then $3x+4x+5x=480$; $12x=480$; $x=40$, etc.

18. Let $a+b$ be the greatest term of the proportion, then $a-b$ is the least.

Also, $\dfrac{a+b}{p+q}=\dfrac{p-q}{a-b}$.

Hence $\dfrac{(a+b)-(p+q)}{p+q}=\dfrac{(p-q)-(a-b)}{a-b}$.

Now $p+q$ is greater than $a-b$.

$\therefore (a+b)-(p+q)$ is greater than $(p-q)-(a-b)$.

And $\therefore (a+b)+(a-b)$ is greater than $(p+q)+(p-q)$.

19. Let x be the rate of the man's rowing, y the rate of the stream, in miles per hour.

Then $x+y=$ the man's rate down stream,

and $x-y=$ the man's rate up stream.

Hence $x+y:x-y=5:3$; $3x+3y=5x-5y$; $4y=x$.

Also $\dfrac{30}{x+y}+\dfrac{30}{x-y}=12$; $\dfrac{30}{5y}+\dfrac{30}{3y}=12$; $16=12y$; $y=1\frac{1}{3}$.

20. Let C contain x pints of brandy and y pints of water.

Then A contains $x+y$ pints of water, and B $x+y$ pints of brandy.

Hence when B and C are mixed, the mixture contains $2x+y$ pints of brandy;

and when A and C are mixed, the mixture contains x pints of brandy;

therefore $2x+y:x=9:1$; $2x+y=9x$; $y=7x$, etc.

21. Let x be the number of quarters; y the price of each in shillings.

Then selling price: $xy=105:100$

$\therefore \dfrac{105xy}{100}=xy+16\times 20$; $\therefore xy=6400$.

Again $x(y+5)=xy+20y$; $\therefore x=4y$.

Hence $4y^2=6400$; $y^2=1600$; $y=40$; $x=160$.

22. Let x be the price of the horse in pounds.

Then since cost price : gain $=100$: gain per cent.

$\therefore x : 144-x=100 : x$

$\therefore x^2+100x=14400$; $x^2+100x+2500=16900$; $x=80$.

23. Let x be the cost of the goods in pounds.

Then $x : 96-x=100 : x$; $x^2+100x=9600$; $x=60$.

24. Let x be the cost of the sheep in pounds.

Then $x : 24-x=100 : x$; $x^2+100x=2400$; $x=20$.

25. Let the first crew row x yards, and the second y yards at each stroke.

Then in 8 minutes the first crew row $320x$ yards,

and in 8 minutes the second crew row $360y$ yards.

Also the second crew has $4x$ yards start.

Hence $320x=360y+4x$; $316x=360y$; $79x=90y$, etc.

26. Let x be the rate of the fast train, y the rate of the slow train.

Then $\frac{180}{y}+\frac{30}{y}=$ time for journey by slow train

$\frac{180}{x}+\frac{15}{y}=$ time for journey by fast train

$\therefore \frac{210}{y} : \frac{180y+15x}{xy}=14:9$; or, $\frac{14}{y} : \frac{12y+x}{xy}=14:9$.

Hence $14\times 9\times x=(12y+x)\times 14$; $8x=12y$; $2x=3y$.

Again $x=y+15$; $\therefore 2y+30=3y$; $y=30$; $x=45$.

27. Let x be the worth of the article, y the selling price, in pounds.

Then $x : x-y=100 : x$

$\therefore x^2=100x-100y$; $x^2-100x+2500=2500-100y$.

To obtain a real value for x, $100y$ must not be greater than 2500 ;

$\therefore y$ cannot be greater than 25.

CXXXIII.

1. Let $A=\frac{m}{B}$ and $B=\frac{n}{C}$.

Then $A=m\div\frac{n}{C}=\frac{m}{n}\cdot C$; $\therefore A \propto C$.

2. Let $A=mB$. Then $\dfrac{A}{P}=m\cdot\dfrac{B}{P}$; $\therefore \dfrac{A}{P}\propto\dfrac{B}{P}$.

3. Let $A=mB$ and $C=nD$. Then $AC=mn.BD$, etc.

4. $5:7=12:x$; $5x=84$, etc.

5. $x:\dfrac{1}{y}=10:\dfrac{1}{2}$; $\dfrac{x}{2}=\dfrac{10}{y}$; $\dfrac{4}{2}=\dfrac{10}{y}$; $y=5$.

6. $x:yz=1:2\times3$; $6x=yz$; $6\times4=y\times2$; $y=12$.

7. $x:\dfrac{y}{z}=6:\dfrac{4}{3}$; $\dfrac{4x}{3}=\dfrac{6y}{z}$; $\dfrac{4x}{3}=\dfrac{6\times5}{7}$; $x=3\dfrac{3}{14}$.

8. $3x+5y:5x+3y=31:25$; $75x+125y=155x+93y$; $32y=80x$, etc.

9. Let $A=mB$, and $B^3=nC^2$; then $\dfrac{A^3}{m^3}=nC^2$; $A^3=\dfrac{n}{m^3}\cdot C^2$;

$$A=\frac{n^{\frac{1}{3}}}{m}\cdot C^{\frac{2}{3}};\ \therefore A\propto C^{\frac{2}{3}}.$$

10. $z:xy=4:2$; $30:3x=2:1$; $x=5$.

11. $A:B=8:12$; $12A=8B$; $A=\dfrac{2}{3}B$.

12. $x^2:y^3=9:64$; $64x^2=9y^3$.

13. $x^2:\dfrac{1}{y^3}=4:\dfrac{1}{27}$; $\dfrac{x^2}{27}=\dfrac{4}{y^3}$; $x^2=\dfrac{108}{y^3}$.

14. $x^3:y^2=27:4$; $4x^3=27y^2$.

15. Let $x=mz$, and $y=\dfrac{n}{z}$; then $xy=mn$; $x=mn.\dfrac{1}{y}$.

16. The area of a triangle $=\frac{1}{2}$(base $\times$ altitude).
Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the triangles; then $a_1b_1=a_2b_2$, or, $a_1:a_2=b_2:b_1$.

17. The area of a parallelogram = base × altitude.

Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the parallelogram

Then $a_1b_1=a_2b_2$; or, $a_1:a_2=b_2:b_1$.

18. Let $y=p+mx+nx^2$.

$$\text{Then } \left.\begin{aligned} 6&=p+m+n\\ 11&=p+2m+4n\\ 18&=p+3m+9n \end{aligned}\right\};\quad \left.\begin{aligned} 5&=m+3n\\ 7&=m+5n \end{aligned}\right\};\ n=1,\ m=2,\ p=3.$$

$\therefore\ y=3+2x+x^2$.

19. $10\times27:9\times9\times10=2\times27:3\times3\times$ required height in feet.

$$\text{Height}=\frac{9\times9\times10\times2\times27}{10\times27\times3\times3}\text{ ft.}=18\text{ ft.}$$

20. Let g, n, l, b represent the area of glass, the number, length, ar breadth of the panes respectively.

Then suppose $g=m.nlb$, where m is a constant,

$n=\frac{p}{b^2}$, where p is a constant.

$l=\frac{q}{b}$, where q is a constant.

$$\text{Then}\qquad g=m.\frac{p}{b^2}.l.\frac{q}{l}=mpq.\frac{1}{b^2}=mpq.\frac{l^2}{q^2}=\frac{mp}{q}.l^2$$

$\therefore\ g$ varies as l^2.

CXXXIV.

1. $a=2,\ d=3,\ n-1=16$

$z=2+16\times3=2+48=50$.

2. $a=4,\ d=4,\ n-1=49$

$z=4+4\times49=4+196=200$.

3. $a=7,\ d=\frac{1}{4},\ n-1=15$

$$z=7+\frac{1}{4}\times15=7+\frac{15}{4}=10\frac{3}{4}.$$

4. $a=\frac{1}{2}, d=-\frac{3}{2}, n-1=22$

$$z=\frac{1}{2}-\frac{3}{2}\times 22=\frac{1}{2}-33=-32\frac{1}{2}.$$

5. $a=\frac{5}{6}, d=\frac{1}{2}-\frac{5}{6}=-\frac{1}{3}, n-1=11$

$$z=\frac{5}{6}-\frac{1}{3}\times 11=\frac{5}{6}-\frac{11}{3}=-2\frac{5}{6}.$$

6. $a=-12, d=4, n-1=13$

$$z=-12+4\times 13=-12+52=40.$$

7. $a=-3, d=8, n-1=15$

$$z=-3+8\times 15=-3+120=117.$$

8. $a=\frac{n-1}{n}, d=\frac{n-2}{n}-\frac{n-1}{n}=\frac{-1}{n}, n-1=n-1$

$$z=\frac{n-1}{n}-\frac{1}{n}\times(n-1)=\frac{n-1}{n}-\frac{n-1}{n}=0.$$

9. $a=x^2+2xy+y^2, d=-2xy, n-1=n-1$

$$z=x^2+2xy+y^2-2xy(n-1)=x^2+y^2-2(n-2)xy.$$

10. $a=\frac{a-b}{a+b}, d=\frac{3a-2b}{a+b}, n-1=n-1$

$$z=\frac{a-b}{a+b}+\frac{3an-2bn-3a+2b}{a+b}=\frac{3an-2bn-2a+b}{a+b}.$$

CXXXV.

1. $2a=4, d=1, n=100$

$$s=\frac{100}{2}\{2+99\times 1\}=50\times 101=5050.$$

2. $2a=4, d=2, n=50$

$$s=25\{4+49\times 2\}=25\times 102=2550.$$

3. $2a=6,\ d=4,\ n=20$

$$s=10\{6+19\times 4\}=10\times 82=820.$$

4. $2a=\frac{1}{2},\ d=\frac{1}{4},\ n=15$

$$s=\frac{15}{2}\left\{\frac{1}{2}+\frac{1}{4}\times 14\right\}=\frac{15}{2}\times 4=30.$$

5. $2a=-18,\ d=2,\ n=12$

$$s=6\{-18+2\times 11\}=6\times 4=24.$$

6. $2a=\frac{5}{3},\ d=-\frac{1}{3},\ n=17$

$$s=\frac{17}{2}\left\{\frac{5}{3}-\frac{1}{3}\times 16\right\}=\frac{17}{2}\times\left(-\frac{11}{3}\right)=-\frac{187}{6}=-31\frac{1}{6}.$$

7. $2a=2,\ d=1,\ n=n$

$$s=\frac{n}{2}\{2+1\times(n-1)\}=\frac{n}{2}.(n+1).$$

8. $2a=2,\ d=3,\ n=n$

$$s=\frac{n}{2}\{2+3(n-1)\}=\frac{n}{2}(3n-1)=\frac{3n^2-n}{2}.$$

9. $2a=2,\ d=7,\ n=n$

$$s=\frac{n}{2}\{2+7(n-1)\}=\frac{n}{2}(7n-5)=\frac{7n^2-5n}{2}.$$

10. $2a=\frac{2(n-1)}{n},\ d=-\frac{1}{n},\ n=n$

$$s=\frac{n}{2}\left\{\frac{2n-2}{n}-\frac{n-1}{n}\right\}=\frac{n}{2}\times\frac{n-1}{n}=\frac{n-1}{2}.$$

CXXXVI.

1. $-14=100+19d$; $19d=-114$; $d=-6.$

2. $-x=x+50d$; $50d=-2x$; $d=-\frac{x}{25}.$

3. $5\frac{1}{2} = -\frac{1}{2} + 48d$; $48d = 6$; $d = \frac{1}{8}$.

4. $-21\frac{3}{4} = -\frac{3}{4} + 24d$; $24d = -21$; $d = -\frac{7}{8}$.

5. $-20 = -10 + 5d$; $5d = -10$; $d = -2$.

6. $0 = 150 + 90d$; $90d = -150$; $d = -\frac{5}{3}$.

CXXXVII.

1. (1.) $\left.\begin{array}{l} a + 58d = 70 \\ a + 65d = 84 \end{array}\right\}$; $7d = 14$; $d = 2$; $a = 70 - 116 = -46$.

(2.) $\left.\begin{array}{l} a + 19d = 93 - 35b \\ a + 20d = 98 - 37b \end{array}\right\}$; $d = 5 - 2b$; $a + 95 - 38b = 93 - 35b$, etc.

(3.) $\left.\begin{array}{l} a + d = \frac{1}{2} \\ a + 54d = 5\cdot 8 \end{array}\right\}$; $\left.\begin{array}{r} 54a + 54d = 27 \\ a + 54d = 5\cdot 8 \end{array}\right\}$; $53a = 21\cdot 2$; $a = \frac{2}{5}$.

(4.) $\left.\begin{array}{l} a + d = 4 \\ a + 86d = -30 \end{array}\right\}$; $\left.\begin{array}{r} 86a + 86d = 344 \\ a + 86d = -30 \end{array}\right\}$; $85a = 374$; $a = 4\cdot 4$.

2. $\left.\begin{array}{l} (a + 2d) + (a + 7d) = 31 \\ (a + 4d) + (a + 9d) = 43 \end{array}\right\}$; $\left.\begin{array}{r} 2a + 9d = 31 \\ 2a + 13d = 43 \end{array}\right\}$; $d = 3$, $a = 2$.

Hence sum of 10 terms $= 5\{4 + 27\} = 155$.

3. $\left.\begin{array}{l} a + (a + 2d) = 0 \\ (a + d) + (a + 6d = 40 \end{array}\right\}$; $\left.\begin{array}{r} 2a + 2d = 0 \\ 2a + 7d = 40 \end{array}\right\}$; $d = 8$, $a = -8$.

Hence sum of 7 terms $= \frac{7}{2}\cdot\{-16 + 48\} = \frac{7}{2} \times 32 = 112$.

4. $\left.\begin{array}{l} a + 3d = 24 \\ a + 4d = 33 \end{array}\right\}$; $d = 9$, $a = -3$.

Hence the 100th term is $-3 + 99 \times 9 = -3 + 891 = 888$.

5. $302=5+(n-1)\times 3$; $300=3n$; $n=100$.

6. $s=\frac{20}{2}\left\{32\frac{1}{6}+19\times 32\frac{1}{6}\right\}=10\times 20\times\frac{193}{6}=6433\frac{1}{3}$.

7. From the formula $s=(a+z)\frac{n}{2}$

$s=(1+103)\times 26s.=2704s.=£135.4s.$

8. (1.) 41st term $=-5+9\times 40=355$

Sum $=(-5+355)\times\frac{41}{2}=7175.$

(2.) 41st term $=4a^2-4a^2\times 40=-156a^2$

Sum $=(4a^2-156a^2)\times\frac{41}{2}=-3116a^2.$

(3.) 41st term $=1+x+(4+2x)40=161+81x$

Sum $=(1+x+161+81x)\times\frac{41}{2}=3321+1681x.$

(4.) 41st term $=-4\frac{1}{2}+3\frac{1}{10}\times 40=119\frac{1}{2}$

Sum $=\left(-4\frac{1}{2}+119\frac{1}{2}\right)\times\frac{41}{2}=2357\frac{1}{2}$.

(5.) 41st term $=\frac{1}{4}+\frac{1}{5}\times 40=8\frac{1}{4}$

Sum $=\left(\frac{1}{4}+8\frac{1}{4}\right)\times\frac{41}{2}=174\frac{1}{4}$.

9. (1.) $a=1002$; $d=-8$; nth term $=a+(n-1)d$

$2=1002-8(n-1)$; $8n=1008$; $n=126$

$s=(10002+2)\times 63=63252.$

(2.) $186=-6+8(n-1)$; $8n=200$; $n=25$

$s=(-6+186)\times\frac{25}{2}=2250.$

(3.) $-72\cdot3x=2\frac{1}{2}x-1\cdot7x(n-1)$; $1\cdot7n=76\cdot5$; $n=45$

$s=(2\cdot5x-72\cdot3x)\times\frac{45}{2}=-1570\cdot5x.$

(4.) $-24=\frac{1}{2}-(n-1)\times\frac{1}{4}$; $\frac{n}{4}=24\frac{3}{4}$; $n=99$

$s=\left(\frac{1}{2}-24\right)\times\frac{99}{2}=-\frac{4653}{4}=-1163\frac{1}{4}.$

(5.) $139(1-m)=m-1+(n-1)2(1-m)$; $139=-1+2(n-1)$; $n=71$

$s=\{m-1+139(1-m)\}\times\frac{71}{2}=4899(1-m).$

(6.) $x-2=x+254-(n-1)\times4$; $4n=2+258$; $n=65$

$s=\{x+254+x-2\}\frac{65}{2}=65x+8190.$

CXXXVIII.

1. $18=3+5d$; $d=3$; means are 6, 9, 12, 15.

2. $-2=2+6d$; $d=-\frac{2}{3}$; means are $1\frac{1}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, $-1\frac{1}{3}$.

3. $\frac{2}{3}=3+4d$; $a=-\frac{7}{12}$; means are $2\frac{5}{12}$, $1\frac{5}{6}$, $1\frac{1}{4}$.

4. $\frac{1}{3}=\frac{1}{2}+5d$; $d=-\frac{1}{30}$; means are $\frac{7}{15}$, $\frac{13}{30}$, $\frac{2}{5}$, $\frac{11}{30}$.

CXXXIX.

1. $n=m+4d$; $d=\frac{n-m}{4}$; means are $\frac{3m+n}{4}$, etc.

2. $m-1=m+1+5d$; $d=-\frac{2}{5}$; means are $\frac{5m+3}{5}$, etc.

3. $n^2+1=n^2+5d$; $d=\frac{1}{5}$; means are $\frac{5n^2+1}{5}$, etc.

4. $x^2-y^2=x^2+y^2+4d$; $d=-\frac{y^2}{2}$; means are $\frac{2x^2+y^2}{2}$, etc.

CXL.

1. $z=1\times 2^6=64$.

2. $z=4\times 3^9=4\times 19683=78732$.

3. $z=5\times 4^8=327680$.

4. $z=8\times\left(\frac{1}{2}\right)^{14}=\frac{2^3}{2^{14}}=\frac{1}{2^{11}}=\frac{1}{2048}$.

5. $z=2\times 3^8=13122$.

6. $z=\frac{1}{64}\times 4^{10}=\frac{4^{10}}{4^3}=4^7=16384$.

7. $z=-\frac{2}{3}\times\left(-\frac{1}{2}\right)^6=\frac{-2}{3\times 2^6}=\frac{-1}{3\times 2^5}=-\frac{1}{96}$.

CXLI.

1. $s=\frac{2(2^{15}-1)}{2-1}=2(32768-1)=65534$.

2. $s=\frac{1(3^6-1)}{3-1}=\frac{729-1}{2}=364$.

3. $s=\frac{a(x^{26}-1)}{x^2-1}$.

4. $s=\frac{a\left(\frac{1}{x^9}-1\right)}{\frac{1}{x}-1}=\frac{a}{x^8}\left(\frac{x^9-1}{x-1}\right)$.

5. $s=\frac{(a^2-x^2)\left\{\frac{1}{(a+x)^7}-1\right\}}{\frac{1}{a+x}-1}=\frac{(a^2-x^2)}{(a+x)^6}\left\{\frac{1-(a+x)^7}{1-a-x}\right\}$

$=\frac{(a-x)\{1-(a+x)^7\}}{(a+x)^5.(1-a-x)}$.

6. $s=\frac{2(3^n-1)}{3-1}=3^n-1$.

7. $s=\frac{7(2^n-1)}{2-1}=7(2^n-1)$.

8. $s=\frac{5\{(-2)^8-1\}}{-2-1}=\frac{5(1-256)}{3}=-425.$

9. $s=\frac{-\frac{2}{3}\left\{\left(-\frac{1}{2}\right)^7-1\right\}}{-\frac{1}{2}-1}=\frac{-\frac{2}{3}\left\{-\frac{1}{128}-1\right\}}{-\frac{3}{2}}=\frac{4}{9}\times\left(-\frac{129}{128}\right)$

$=-\frac{43}{96}.$

CXLII.

1. $s=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2.$

2. $s=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}.$

3. $s=\frac{3}{1-\frac{1}{9}}=\frac{3\times 9}{8}=\frac{27}{8}.$

4. $s=\frac{\frac{2}{3}}{1-\frac{1}{2}}=\frac{4}{3}.$

5. $s=\frac{\frac{3}{4}}{1-\frac{1}{3}}=\frac{3\times 3}{4\times 2}=1\frac{1}{8}.$

6. $s=\frac{\frac{1}{2}}{1+\frac{2}{3}}=\frac{1\times 3}{2\times 5}=\frac{3}{10}.$

7. $s=\frac{8}{1-\frac{1}{12}}=\frac{8\times 12}{11}=8\frac{8}{11}.$

8. $s=\frac{1\cdot 5}{1-\frac{1}{3}}=\frac{1\cdot 5\times 3}{2}=2\frac{1}{4}.$

9. $s=\frac{64}{1-\frac{1}{4}}=\frac{64\times 4}{3}=85\frac{1}{3}.$

10. $s=\frac{2x^3}{1+\frac{1}{8x^2}}=\frac{16x^5}{8x^2+1}.$

11. $s=\frac{a}{1-\frac{b}{a}}=\frac{a^2}{a-b}.$

12. $s=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{1}{9}.$

13. $s = \dfrac{x}{1+\dfrac{y}{x}} = \dfrac{x^2}{x+y}$.

14. $s = 86\left\{\dfrac{\dfrac{1}{100}}{1-\dfrac{1}{100}}\right\} = 86 \times \dfrac{1}{99} = \dfrac{86}{99}$.

15. $s = \dfrac{5}{10} + \dfrac{4}{100} + \dfrac{4}{1000} + \ldots = \dfrac{5}{10} + \dfrac{\dfrac{4}{100}}{1-\dfrac{1}{10}} = \dfrac{5}{10} + \dfrac{4}{90} = \dfrac{49}{90}$.

16. $s = \dfrac{8}{10} + \dfrac{36}{1000} + \dfrac{36}{100000} + \ldots = \dfrac{8}{10} + \dfrac{\dfrac{36}{1000}}{1-\dfrac{1}{100}} = \dfrac{8}{10} + \dfrac{36}{990} = \dfrac{46}{55}$.

CXLIII.

1. $243 = 3f^4$; $f^4 = 81$; $f = 3$, etc.

2. $1024 = 1 \times f^5$; $f^5 = 4^5$; $f = 4$, etc.

3. $16 = 1 \times f^4$; $f^4 = 16$; $f = 2$, etc.

4. $\dfrac{243}{64} = \dfrac{1}{2} \times f^5$; $f^5 = \dfrac{243}{32}$; $f = \dfrac{3}{2}$, etc.

CXLIV.

1. (1.) $s = \{16 + 7(12-1)\} \times \dfrac{12}{2} = 93 \times 6 = 558$.

(2.) $s = \{232 - 8(10-1)\} \times \dfrac{10}{2} = 160 \times 5 = 800$.

(3.) $s = \dfrac{3}{1-\dfrac{1}{6}} = \dfrac{18}{5}$.

(4.) $s=\dfrac{2}{1+\dfrac{1}{8}}=\dfrac{16}{9}$.

(5.) $s=\left\{1-\dfrac{7}{6}(13-1)\right\}\times\dfrac{13}{2}=(-13)\times\dfrac{13}{2}=-\dfrac{169}{2}$.

(6.) $s=\dfrac{\dfrac{1}{2}\cdot\left\{\dfrac{2^6}{3^6}-1\right\}}{-\dfrac{2}{3}-1}=\dfrac{3}{10}\cdot\left(\dfrac{3^6-2^6}{3^6}\right)=\dfrac{665}{2430}=\dfrac{133}{486}$.

(7.) $s=\left\{1-\dfrac{3}{2}\times 28\right\}\times\dfrac{29}{2}=(-41)\times\dfrac{29}{2}=-\dfrac{1189}{2}$.

(8.) $s=\left\{\dfrac{10}{7}+\dfrac{2}{7}\times 7\right\}\times\dfrac{8}{2}=\dfrac{24}{7}\times 4=13\dfrac{5}{7}$.

(9.) $s=\dfrac{\dfrac{1}{3}}{1-\dfrac{2}{3}}=\dfrac{1}{3}\times\dfrac{3}{1}=1$.

(10.) $s=\left\{\dfrac{6}{5}-2\times 9\right\}\times\dfrac{10}{2}=\left(-\dfrac{84}{5}\right)\times 5=-84$.

(11.) The common factor $=-\sqrt{6}\div\dfrac{\sqrt{3}}{\sqrt{5}}=-\dfrac{\sqrt{30}}{\sqrt{3}}=-\sqrt{10}$

$$s=\frac{\sqrt{\dfrac{3}{5}}\{(-\sqrt{10})^8-1\}}{-\sqrt{10}-1}=\frac{\sqrt{3}.\{10000-1\}}{\sqrt{5}\{-\sqrt{10}-1\}}=-\frac{9999\sqrt{3}}{\sqrt{5}(\sqrt{10}+1)}.$$

(12.) $s=\dfrac{-\dfrac{7}{5}\left\{-\dfrac{5^5}{2^5}-1\right\}}{-\dfrac{5}{2}-1}=-\dfrac{2}{5}\left(\dfrac{5^5+2^5}{2^5}\right)=-\dfrac{3157}{80}$.

2. Let the series be a, af, af^2, af^3, af^4.
Then $a\times af\times af^2\times af^3\times af^4=32$; $a^5f^{10}=2^5$; $af^2=2$.

3. $b=\dfrac{a+c}{2}$ and $b'=\sqrt{(ac)}$

$\therefore \dfrac{b}{b'}=\dfrac{a+c}{2\sqrt{(ac)}}$.

4. The arithmetic mean is $\frac{a+b}{2}$; the geometric is $\sqrt{(ab)}$.

Now since the square of every number is *positive*

$(\sqrt{a}-\sqrt{b})^2$ is greater than 0

$a-2\sqrt{(ab)}+b$ is greater than 0.

$\therefore \frac{a+b}{2}$ is greater than $\sqrt{(ab)}$.

5. $a+(a+d)+(a+2d)=12$; $3a+3d=12$; $a+d=4$.
Also, $a+5d=12$. Hence $4d=8$; $d=2$; $a=2$.
Then sum of 6 terms $=\{4+10\}\times 3=42$.

6. Let f be the common factor: $af=b$ and $af^2=c$
$\therefore ac=a^2f^2=b^2$.

7. $2n\times\frac{1}{2n}=x^2$; $1=x^2$; $x=\pm 1$.

8. $2n+\frac{1}{2n}=2y$; $y=n+\frac{1}{4n}$.

9. The sum of the geometric progression is $\frac{3^4-1}{3-1}$ or $\frac{80}{2}$ or 40.

The sum of the arithmetic progression is $\{8+4(n-1)\}\frac{n}{2}$.

Hence $2n+2n^2=40$; $n^2+n=20$; whence $n=4$.

10. The first term is 1, the constant difference 1.

Hence $153=\{2+(7+n-1)\}\frac{7+n}{2}$; $306=(8+n)(7+n)$;

$n^2+15n=250$; $n=10$.

11. Let n be the number of terms.

Then $\{2+2(n-1)\}\frac{n}{2}=2n\times\frac{n}{2}=n^2$.

12. Let the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$.
Then $5a+10d=95$; and $\therefore a+2d=19$.

13. $22=\left\{6\frac{2}{3}+\frac{13}{9}(n-1)\right\}\frac{n}{2}$; $44=n\left(\frac{47+13n}{9}\right)$;

$396=47n+13n^2$; whence $n=4$.

14. Let $100x+10y+z$ be the number.

Then $x+z=2y$ (1)

and $\frac{100x+10y+z}{y+z}=107$; or $100x-97y-106z=0$ (2)

and $100x+10y+z-396=100z+10y+x$. (3)

From (3) $99x-396=99z$; or $x-z=4$.

From this and (1) $x=y+2$ and $z=y-2$.

Hence from (2) $100y+200-97y-106y+212=0$.

Thus $y=4$, $x=6$, $z=2$.

15. In any geometrical series a, af, etc.

the $(p+q)$th term is af^{p+q-1}

the $(p-q)$th term is af^{p-q-1}

the pth term is af^{p-1}.

Hence $mn=af^{p+q-1}\times af^{p-q-1}=a^2f^{2p-2}=(af^{p-1})^2$

$\therefore af^{p-1}=\sqrt{(mn)}$.

16. Let x and y be the numbers. Then

$$\left.\begin{array}{l} x-y=48 \\ \frac{x+y}{2}=\sqrt{(xy)}+18 \end{array}\right\};\quad \left.\begin{array}{l} x-y=48 \\ x+y-2\sqrt{xy}=36 \end{array}\right\};\quad \left.\begin{array}{r} x-y=48 \\ \sqrt{x}-\sqrt{y}=\ \ 6 \end{array}\right\};$$

dividing, $\sqrt{x}+\sqrt{y}=8$; hence $2\sqrt{x}=14$; $\sqrt{x}=7$; $x=49$; $y=1$

17. $11=1+4d$; $4d=10$; $d=\frac{5}{2}$, etc.

18. $2\cdot748=\{\cdot068+(n-1)\times\cdot0004\}\times\frac{n}{2}$; $5\cdot496=\cdot0676n+\cdot0004n^2$;

$54960=4n^2+676n$; $n^2+169n=13740$; $n=60$.

19. $-1=1+10d$; $10d=-2$; $d=-\frac{1}{5}$, etc.

20. The $(n+1)$th term is 2^n.

The sum of n terms is $\dfrac{2^n-1}{2-1}=2^n-1$.

21. Sum of the *first* p terms $=\dfrac{a(r^p-1)}{r-1}$.

Sum of the *second* p terms $=\dfrac{ar^p(r^p-1)}{r-1}$.

Sum of the *third* p terms $=\dfrac{ar^{2p}(r^p-1)}{r-1}$, and so on, the sums forming a geometric series whose common factor is r^p.

22. $\left.\begin{array}{l}(a-2d)+(a-d)+a+(a+d)+(a+2d)=55\\(a-2d)^2+(a-d)^2+a^2+(a+d)^2+(a+2d)^2=765\end{array}\right\}$;

$\left.\begin{array}{l}5a=55\\5a^2+10d^2=765\end{array}\right\}$; $a=11$; $d=4$, etc.

23. Let the series be a, af, af^2, af^3, af^4.

$\left.\begin{array}{l}af^4-a:af^3-af=10:3\\af+af^3=2a^2f\end{array}\right\}$; $\left.\begin{array}{l}3f^4-3=10f^3-10f\\1+f^2=2a\end{array}\right\}$;

$\left.\begin{array}{l}3(f^4-1)=10f(f^2-1)\\1+f^2=2a\end{array}\right\}$; $\left.\begin{array}{ll}3(f^2+1)=10f & (1)\\1+f^2=2a & (2)\end{array}\right\}$;

from (1) we find $f=3$; and then from (2) $a=5$, etc.

24. This is explained in Art. 472.

25. Let $100x+10y+z$ be the number. Then

$$\left.\begin{array}{l}y^2=xz\\x+y+z=13\\100x+10y+z+792=100z+10y+x\end{array}\right\}$$

From the third equation we get $x-z=-8$.

Hence $x^2-2xz+z^2=64$.

But $\quad 4xz \quad =4y^2$

$\therefore (x+z)^2=64+4y^2$.

Also $x+z=13-y$

$\therefore (13-y)^2=64+4y^2$.

Hence we get $y=3$; and then $x=1$ and $z=9$.

26. Let x be the increase per cent. Then the population of any year is $\frac{100+x}{100}$ times the population of the preceding year.

Hence $10000 \times \left(\frac{100+x}{100}\right)^4 = 14641$.

Take the fourth root of each side; then

$$10 \times \frac{100+x}{100} = 11;\ 100+x=110;\ x=10.$$

CXLV.

1. First insert two arithmetic means between $\frac{1}{6}$ and $\frac{1}{24}$.

$$\frac{1}{24}=\frac{1}{6}+3d;\ 3d=-\frac{3}{24};\ d=-\frac{1}{24}.$$

Hence the arithmetic means are $\frac{1}{8}$ and $\frac{1}{12}$

$\therefore$ the harmonic means are 8 and 12.

2. $\frac{1}{3}=\frac{1}{2}+5d;\ 5d=-\frac{1}{6};\ d=-\frac{1}{30}.$

Hence the arithmetic means are $\frac{7}{15}, \frac{13}{30}, \frac{2}{5}, \frac{11}{30}$

$\therefore$ the harmonic means are $\frac{15}{7}, \frac{30}{13}, \frac{5}{2}, \frac{30}{11}$.

3. $\frac{2}{3}=3+4d;\ 4d=-\frac{7}{3};\ d=-\frac{7}{12}.$

Hence the arithmetic means are $\frac{29}{12}, \frac{11}{6}, \frac{5}{4}$

$\therefore$ the harmonic means are $\frac{12}{29}, \frac{6}{11}, \frac{4}{5}$.

4. $18=3+5d;\ 5d=15;\ d=3.$

Hence the arithmetic means are 6, 9, 12, 15

$\therefore$ the harmonic means are $\frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}$.

5. $2^{-1}=\frac{1}{2}$; and we have to insert five arithmetic means between -1 and 2.

$2=-1+6d$; $6d=3$; $d=\frac{1}{2}$.

Hence the arithmetic means are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$

$\therefore$ the harmonic means are -2, ∞, 2, 1, $\frac{2}{3}$.

6. $-2=2+6d$; $6d=-4$; $d=-\frac{2}{3}$.

Hence the arithmetic means are $\frac{4}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, $-\frac{4}{3}$

$\therefore$ the harmonic means are $\frac{3}{4}$, $\frac{3}{2}$, ∞, $-\frac{3}{2}$, $-\frac{3}{4}$.

7. $\frac{23}{6}=\frac{1}{3}+7d$; $7d=\frac{21}{6}$; $d=\frac{1}{2}$.

Hence the arithmetic means are $\frac{5}{6}$, $\frac{4}{3}$, $\frac{11}{6}$, $\frac{7}{3}$, $\frac{17}{6}$, $\frac{10}{3}$

$\therefore$ the harmonic means are $\frac{6}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.

8. $\frac{1}{3y}=\frac{1}{2x}+(n+1)d$; $d=\frac{2x-3y}{6xy(n+1)}$.

First arithmetic mean is $\frac{1}{2x}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+3y+2x-3y}{6xy(n+1)}$

$=\frac{3ny+2x}{6xy(n+1)}$.

Second arithmetic mean is $\frac{3ny+2x}{6xy(n+1)}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+4x-3y}{6xy(n+1)}$.

. . . .

nth arithmetic mean is $\frac{1}{3y}-\frac{2x-3y}{6xy(n+1)}=\frac{2nx+3y}{6xy(n+1)}$.

Hence the harmonic means are $\frac{6xy(n+1)}{3ny+2x}$, $\frac{6xy(n+1)}{3ny+4x-3y}$, . . .

$\frac{6xy(n+1)}{2nx+3y}$.

9. Let $\frac{1}{x}$ and $\frac{1}{y}$ be the second and third terms.

Then 2, x, y are in arithmetic progression.

$$\left.\begin{aligned}&\text{Hence } 2+y=2x\\ &\text{and } \frac{1}{2}+\frac{1}{x}+\frac{1}{y}=\frac{11}{12}\end{aligned}\right\}; \ \frac{1}{2}+\frac{1}{x}+\frac{1}{2x-2}=\frac{11}{12}$$

$$\frac{2x-2+x}{2x^2-2x}=\frac{5}{12}; \ 36x-24=10x^2-10x; \ 10x^2-46x=-24;$$

hence we find $x=4$ or $\frac{3}{5}$; and $\therefore y=6$ or $-\frac{4}{5}$.

The arithmetic progressions will therefore be

$$-4, \ -2, \ 0, \ 2, \ 4, \ 6, \text{ and } \frac{31}{5}, \ \frac{24}{5}, \ \frac{17}{5}, \ 2, \ \frac{3}{5}, -\frac{4}{5}.$$

$\therefore$ the harmonic progressions are

$$-\frac{1}{4}, \ -\frac{1}{2}, \ \infty, \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{6} \text{ and } \frac{5}{31}, \ \frac{5}{24}, \ \frac{5}{17}, \ \frac{1}{2}, \ \frac{5}{3}, \ -\frac{5}{4}.$$

10. Let x and y be the numbers. Then

$$\frac{x+y}{2}=\sqrt{(xy)}+13 \ (1); \text{ and } \sqrt{(xy)}=\frac{2xy}{x+y}+12$$

$$\frac{2xy}{x+y}=\sqrt{(xy)}-12 \ (2).$$

Multiply (1) by (2); $xy=xy+\sqrt{(xy)}-156$.

Hence $\sqrt{(xy)}=156$; from this and (1) we find $x=104$, $y=234$.

11. $2b=a+c$ and $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$

$$\therefore 2b\left(\frac{1}{b}+\frac{1}{d}\right)=\frac{2}{c}(a+c); \ 2+\frac{2b}{d}=\frac{2a}{c}+2;$$

$\therefore bc=ad$, and $\therefore a:b=c:d$.

12. $$x=\frac{2mn}{m+n}; \ \frac{1}{x-m}+\frac{1}{x-n}=\frac{1}{\frac{2mn}{m+n}-m}+\frac{1}{\frac{2mn}{m+n}-n}$$

$$=\frac{m+n}{mn-m^2}+\frac{m+n}{mn-n^2}=\frac{m+n}{n-m}\left\{\frac{1}{m}-\frac{1}{n}\right\}=\frac{m+n}{mn}=\frac{1}{m}+\frac{1}{n}$$

13. $\left.\begin{array}{l} x+y+z=11 \\ x^2+y^2+z^2=49 \\ y=\dfrac{2xz}{x+z} \end{array}\right\}; \quad \left.\begin{array}{l} (x+z)^2=121-22y+y^2 \\ x^2+z^2=49-y^2 \\ 2xz=72-22y+2y^2 \end{array}\right\}$

Now $(x+z)y=2xz$.

$\therefore\ (11-y)y=72-22y+2y^2$; whence $y=3$.

Then $x+z=8$, and $xz=12$; whence $x=2$, $z=6$.

14. $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are the pth, qth, and rth terms of an arithmetic progression, suppose a, $a+d$. . . .

Then $\left.\begin{array}{l} \dfrac{1}{x}=a+(p-1)d \\ \dfrac{1}{y}=a+(q-1)d \\ \dfrac{1}{z}=a+(r-1)d \end{array}\right\}; \quad \begin{array}{l} \dfrac{1}{x}-\dfrac{1}{z}=(p-r)d \\ \dfrac{1}{z}-\dfrac{1}{y}=(r-q)d \\ \dfrac{1}{y}-\dfrac{1}{x}=(q-p)d \end{array}$

Hence $\left.\begin{array}{l} z-x=(p-r)xzd \\ y-z=(r-q)yzd \\ x-y=(q-p)xyd \end{array}\right\};$

adding $0=(p-r)xzd+(r-q)yzd+(q-p)xyd$

$\therefore\quad 0=(p-r)xz+(r-q)yz+(q-p)xy.$

15. $\dfrac{2ab}{a+b}, \dfrac{2bc}{b+c}, \dfrac{2ca}{c+a}$ are in A. P.

$\therefore\ \dfrac{bc}{b+c}-\dfrac{ab}{a+b}=\dfrac{ca}{c+a}-\dfrac{bc}{b+c}$

$\therefore\ \dfrac{b^2c-ab^2}{(b+c)(a+b)}=\dfrac{c^2a-bc^2}{(b+c)(c+a)}$, or, $\dfrac{b^2(c-a)}{a+b}=\dfrac{c^2(a-b)}{c+a}$

$\therefore\ b^2(c^2-a^2)=c^2(a^2-b^2)$, or, $2b^2c^2=a^2b^2+a^2c^2$

$\therefore\ \dfrac{2}{a^2}=\dfrac{1}{c^2}+\dfrac{1}{b^2}$; $\because\ b^2, a^2, c^2$ are in H. P.

Again, $\dfrac{a+b}{2ab}, \dfrac{b+c}{2bc}, \dfrac{c+a}{2ca}$ are in A. P.

$$\therefore \frac{b+c}{bc}-\frac{a+b}{ab}=\frac{c+a}{ac}-\frac{b+c}{bc}$$

$$\therefore \frac{1}{c}+\frac{1}{b}-\frac{1}{b}-\frac{1}{a}=\frac{1}{a}+\frac{1}{c}-\frac{1}{c}-\frac{1}{b}$$

$$\therefore \frac{1}{c}+\frac{1}{b}=\frac{2}{a};\ \therefore b,\ a,\ c \text{ are in H. P.}$$

16. (1.) When c is the arithmetic mean between a and b.

$$\frac{\frac{a+b}{2}+2a}{\frac{a+b}{2}-b}+\frac{\frac{a+b}{2}+2b}{\frac{a+b}{2}-a}=\frac{5a+b}{a-b}+\frac{a+5b}{b-a}$$

$$=\frac{5a+b-a-5b}{a-b}=\frac{4a-4b}{a-b}=4.$$

(2.) When c is the geometric mean between a and b.

$$\frac{\sqrt{(ab)}+2a}{\sqrt{(ab)}-b}+\frac{\sqrt{(ab)}+2b}{\sqrt{(ab)}-a}$$

$$=\frac{\sqrt{a}(\sqrt{b}+2\sqrt{a})}{\sqrt{b}(\sqrt{a}-\sqrt{b})}-\frac{\sqrt{b}(\sqrt{a}+2\sqrt{b})}{\sqrt{a}(\sqrt{a}-\sqrt{b})}=\frac{a\sqrt{b}+2a\sqrt{a}-b\sqrt{a}-2b\sqrt{b}}{\sqrt{(ab)}.(\sqrt{a}-\sqrt{b})}$$

$$=\frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}};$$ and since $a+b$ is greater than $2\sqrt{(ab)}$,

$$\therefore \frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}} \text{ is greater than } \frac{4\sqrt{(ab)}+3\sqrt{(ab)}}{\sqrt{(ab)}}, \text{ or, than } 7.$$

(3.) When c is the harmonic mean between a and b.

$$\frac{\frac{2ab}{a+b}+2a}{\frac{2ab}{a+b}-b}+\frac{\frac{2ab}{a+b}+2b}{\frac{2ab}{a+b}-a}=\frac{4ab+2a^2}{ab-b^2}+\frac{4ab+2b^2}{ab-a^2}$$

$$=\frac{4a^2b+2a^3-4ab^2-2b^3}{ab(a-b)}=\frac{2a^2+6ab+2b^2}{ab}.$$

Now a^2+b^2 is greater than $2ab$

$$\therefore \frac{2a^2+2b^2+6ab}{ab} \text{ is greater than } \frac{4ab+6ab}{ab}, \text{ or, than } 10.$$

CXLVI.

1. $n.(n-1)=12\times 11=132.$

2. $n.(n-1).(n-2)=16\times 15\times 14=3360.$

3. $n.(n-1).(n-2).(n-3)=20\times 19\times 18\times 17=116280.$

4. $n.(n-1).(n-2)(n-3)(n-4)=8\times 7\times 6\times 5\times 4=6720.$

5. Number of letters is 11, and three are twice repeated,

$\therefore$ number of permutations $=\dfrac{\underline{|11}}{\underline{|2}.\underline{|2}.\underline{|2}}=4989600.$

6. $n.(n-1)\ .\ .\ .\ (n-7)=8\times 7\times 6\times 5\times 4\times 3\times 2\times 1=40320.$

7. $n.(n-1)\ .\ .\ .\ (n-9)=\underline{|10}=3628800.$

8. (1.) The number of signals in which we have 3 flags of different colours is $5.4.3=60.$

(2.) When 2 flags in a signal are of the same colour, suppose red, we can put 4 different colours with these, and each of the resulting signals may be arranged in 3 different ways, that is, with both the red first, or both last, or one first and one last. Thus we shall have 12 different signals with two reds. Similarly we shall have 12 different signals with two of each of the other 4 colours.

$\therefore$ we shall have 5×12 or 60 in all.

(3.) When we have 3 of the same colour, we shall have 5 different signals in all.

$\therefore$ total number $=60+60+5=125.$

9. Number of permutations $=\dfrac{|7}{\underline{|2}}=\dfrac{5040}{2}=2520.$

10. $n:n(n-1)(n-2)=1:20;$

$20n=n(n-1)(n-2);\ n^2-3n+2=20;\ n=6.$

11. $m.(m-1)(m-2):(m+2)(m+1)m=1:5$;
$5m.(m-1)(m-2)=(m+2)(m+1)m$;
$5m^2-15m+10=m^2+3m+2$; $m=4$.

12. Number of letters is 7, and therefore the number of permutations in which *cd* stand first $=\underline{|5}=120$.

13. Number of letters is 9, one of which is repeated twice, one three times, and one four times.

$\therefore$ Number of permutations $=\dfrac{\underline{|9}}{\underline{|2}.\underline{|3}.\underline{|4}}=9\times4\times7\times5=1260$.

14. In *Conceit* we have 7 letters, one of which occurs twice,

$\therefore$ number of permutations $=\dfrac{\underline{|7}}{\underline{|2}}=2520$.

In *Talavera*, 8 letters, one of which is repeated three times,

$\therefore$ number of permutations $=\dfrac{\underline{|8}}{\underline{|3}}=6720$.

In *Calcutta*, 8 letters, 3 of them repeated twice,

$\therefore$ number of permutations $=\dfrac{\underline{|8}}{2\times2\times2}=5040$.

In *Proposition*, 11 letters, 2 repeated twice, 1 thrice,

$\therefore$ number of permutations $=\dfrac{\underline{|11}}{\underline{|2}.\ \underline{|2}.\ \underline{|3}}=1663200$.

In *Mississippi*, 11 letters, 2 repeated 4 times, 1 twice,

$\therefore$ number of permutations $=\dfrac{\underline{|11}}{\underline{|4}.\ \underline{|4}.\ \underline{|2}}=34650$.

CXLVII.

1. ${}_{100}C_4=\dfrac{100\times99\times98\times97}{1\times2\times3\times4}=3921225$.

2. ${}_6C_5=\dfrac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}=6$.

3. Leave out a, and find how many combinations can be formed out of the remaining 9 letters, 4 at a time

$${}_9C_4=\frac{9\times8\times7\times6\times5}{1\times2\times3\times4\times5}=126,$$ with each of which a will combine to form a word of 5 letters.

4. Out of the 19 consonants we get $\frac{19\times18\times17}{1\times2\times3}$ combinations of 3,

then $\frac{19\times18\times17}{1\times2\times3}\times5$ = number of ways in which we can get 3 consonants and 1 vowel, and each combination of 4 letters admits of $\underline{|4}$ permutations.

$\therefore$ number of words $=\frac{19.18.17}{1.2.3}\times5\times\underline{|4}=116280.$

5. $\frac{n.(n-1)(n-2)(n-3)}{1.2.3.4}:\frac{n.(n-1)}{1.2}=15:2$; $(n-2)(n-3)=90$;

$n=12.$

6. $\frac{n.(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}=\frac{18}{5}\times\frac{n.(n-1)(n-2)}{1.2.3}$;

$(n-3)(n-4)=18\times4$; $n=12.$

7. Number of words $=\frac{17.16.15}{1.2.3}\times\frac{5.4}{1.2}\times\underline{|5}=816000.$

8. Number $=\frac{12.11.10.9.8.7}{1.2.3.4.5.6}\times\frac{5.4.3}{1.2.3}\times\underline{|9}=3353011200.$

9. $n(n-1)(n-2)=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\times6$; $6(n-3)=24$;

$n-3=4$; $n=7.$

10. Taking the coins singly, by twos, by threes, and so on,

number of combinations $=6+\frac{6.5}{1.2}+\frac{6.5.4}{1.2.3}+\frac{6.5.4.3}{1.2.3.4}+\frac{6.5.4.3.2}{1.2.3.4.5}+1$

$=6+15+20+15+6+1=63.$

11. $\frac{n.(n-1)(n-2)}{1.2.3}=425\times n$; $(n-1)(n-2)=2550$; hence $n=52.$

12. Number $=\frac{12.11.10}{1.2.3}\times\frac{16.15.14}{1.2.3}=123200.$

13. (1.) The number is $\frac{36.35.34.33.32}{1.2.3.4.5}=376992.$

(2.) He will go out with as many different parties as can be formed by taking 4 out of 35.

$\therefore$ he goes with $\frac{35.34.33.32}{1.2.3.4}=52360.$

CXLVIII.

1. $a^4+4a^3x+\frac{4.3}{1.2}a^2x^2+\frac{4.3.2}{1.2.3}ax^3+x^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$

2. $b^6+6b^5c+\frac{6.5}{1.2}b^4c^2+\frac{6.5.4}{1.2.3}b^3c^3+\frac{6.5.4.3}{1.2.3.4}b^2c^4+\frac{6.5.4.3.2}{1.2.3.4.5}bc^5+c^6$
$=b^6+6b^5c+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6.$

3. $a^7+7a^6b+\frac{7.6}{1.2}a^5b^2+\frac{7.6.5}{1.2.3}a^4b^3+\frac{7.6.5.4}{1.2.3.4}a^3b^4+\frac{7.6.5.4.3}{1.2.3.4.5}a^2b^5$
$+\frac{7.6.5.4.3.2}{1.2.3.4.5.6}ab^6+b^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$
$+21a^2b^5+7ab^6+b^7.$

4. $x^8+8x^7y+\frac{8.7}{1.2}x^6y^2+\frac{8.7.6.}{1.2.3.}x^5y^3+\frac{8.7.6.5}{1.2.3.4}x^4y^4+\frac{8.7.6.5.4}{1.2.3.4.5}x^3y^5$
$+\frac{8.7.6.5.4.3}{1.2.3.4.5.6}x^2y^6+\frac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}xy^7+y^8$
$=x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8.$

5. $5^4+4.5^3.4a+\frac{4.3}{1.2}.5^2.4^2a^2+\frac{4.3.2}{1.2.3}.5.4^3a^3+(4a)^4$
$=625+2000a+2400a^2+1280a^3+256a^4.$

6. $(a^2)^5+5(a^2)^4.bc+\frac{5.4}{1.2}(a^2)^3.(bc)^2+\frac{5.4.3}{1.2.3}(a^2)^2.(bc)^3+\frac{5.4.3.2}{1.2.3.4}a^2(bc)^4$
$+(bc)^5=a^{10}+5a^8bc+10a^6b^2c^2+10a^4b^3c^3+5a^2b^4c^4+b^5c^5.$

CXLIX.

1. $a^6 - 6a^5x + \frac{6.5}{1.2}.a^4x^2 - \frac{6.5.4}{1.2.3}a^3x^3 + \frac{6.5.4.3}{1.2.3.4}a^2x^4 - \frac{6.5.4.3.2}{1.2.3.4.5}ax^5 + x^6$

$= a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6.$

2. $b^7 - 7b^6c + \frac{7.6}{1.2}b^5c^2 - \frac{7.6.5}{1.2.3}b^4c^3 + \frac{7.6.5.4}{1.2.3.4}.b^3c^4 - \frac{7.6.5.4.3}{1.2.3.4.5}b^2c^5$

$+ \frac{7.6.5.4.3.2}{1.2.3.4.5.6}.bc^6 - c^7$

$= b^7 - 7b^6c + 21b^5c^2 - 35b^4c^3 + 35b^3c^4 - 21b^2c^5 + 7bc^6 - c^7.$

3. $(2x - 3y)^5 = \left\{ 2x\left(1 - \frac{3y}{2x}\right) \right\}^5$

$= 32x^5 \left\{ 1 - 5.\frac{3y}{2x} + \frac{5.4}{1.2} \cdot \left(\frac{3y}{2x}\right)^2 - \frac{5.4.3}{1.2.3} \cdot \left(\frac{3y}{2x}\right)^3 \right.$

$\left. + \frac{5.4.3.2}{1.2.3.4} \cdot \left(\frac{3y}{2x}\right)^4 - \left(\frac{3y}{2x}\right)^5 \right\}$

$= 32x^5 \left\{ 1 - \frac{15}{2} \cdot \frac{y}{x} + \frac{5.2.9}{4} \cdot \frac{y^2}{x^2} - \frac{5.2.27}{8} \cdot \frac{y^3}{x^3} \right.$

$\left. + \frac{5.81}{16} \cdot \frac{y^4}{x^4} - \frac{243}{32} \cdot \frac{y^5}{x^5} \right\}$

$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5.$

4. $1 - 5(2x) + \frac{5.4}{1.2}(2x)^2 - \frac{5.4.3}{1.2.3}(2x)^3 + \frac{5.4.3.2}{1.2.3.4}(2x)^4 - (2x)^5$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5.$

5. $1 - 10x + \frac{10.9}{1.2}x^2 - \frac{10.9.8}{1.2.3}x^3 + \frac{10.9.8.7}{1.2.3.4}.x^4 - \frac{10.9.8.7.6}{1.2.3.4.5}x^5$

$+ \frac{10.9.8.7}{1.2.3.4}x^6 - \frac{10.9.8}{1.2.3}x^7 + \frac{10.9}{1.2} \cdot x^8 - 10x^9 + x^{10}$

$= 1 - 10x + 45x^2 - 120x^3 + 210x^4 - 252x^5 + 210x^6 - 120x^7$

$+ 45x^8 - 10x^9 + x^{10}.$

6. $a^{24} - 8a^{21}b^2 + \frac{8.7}{1.2}a^{18}b^4 - \frac{8.7.6}{1.2.3}a^{15}b^6 + \frac{8.7.6.5}{1.2.3.4}a^{12}b^8 - \frac{8.7.6}{1.2.3}a^9b^{10}$

$+\frac{8.7}{1.2}a^6b^{12} - 8a^3b^{14} + b^{16}$

$= a^{24} - 8a^{21}b^2 + 28a^{18}b^4 - 56a^{15}b^6 + 70a^{12}b^8 - 56a^9b^{10}$

$+ 28a^6b^{12} - 8a^3b^{14} + b^{16}.$

CL.

1. $\{(a+2b) - c\}^3 = (a+2b)^3 - 3(a+2b)^2.c + 3(a+2b).c^2 - c^3$

$= a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c + 3ac^2 + 6bc^2 - c^3.$

2. $\{(1-2x) + 3x^2\}^3 = (1-2x)^3 + 3(1-2x)^2.\ 3x^2 + 3(1-2x).9x^4 + 27x^6$

$= 1 - 6x + 12x^2 - 8x^3 + 9x^2 - 36x^3 + 36x^4 + 27x^4 - 54x^5 + 27x^6$

$= 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6.$

3. $(x^3 - x^2 + x)^3 = x^3(x^2 - x + 1)^3 = x^3\{(1-x) + x^2\}^3$

$= x^3\{(1-x)^3 + 3(1-x)^2.x^2 + 3(1-x).x^4 + x^6\}$

$= x^3\{1 - 3x + 3x^2 - x^3 + 3x^2 - 6x^3 + 3x^4 + 3x^4 - 3x^5 + x^6\}$

$= x^9 - 3x^8 + 6x^7 - 7x^6 + 6x^5 - 3x^4 + x^3.$

4. $\{(1+2x^{\frac{1}{6}}) + 3x^{\frac{1}{3}}\}^3 = (1+2x^{\frac{1}{6}})^3 + 3(1+2x^{\frac{1}{6}})^2.3x^{\frac{1}{3}} + 3(1+2x^{\frac{1}{6}}).9x^{\frac{2}{3}} + 27x$

$= 1 + 6x^{\frac{1}{6}} + 12x^{\frac{1}{3}} + 8x^{\frac{1}{2}} + 9x^{\frac{1}{3}} + 36x^{\frac{1}{2}} + 36x^{\frac{2}{3}} + 27x^{\frac{2}{3}} + 54x^{\frac{5}{6}} + 27x$

$= 27x + 54x^{\frac{5}{6}} + 63x^{\frac{2}{3}} + 44x^{\frac{1}{2}} + 21x^{\frac{1}{3}} + 6x^{\frac{1}{6}} + 1.$

5. $\left\{(x+1) - \frac{1}{x}\right\}^3 = (x+1)^3 - 3(x+1)^2.\frac{1}{x} + 3(x+1).\frac{1}{x^2} - \frac{1}{x^3}$

$= x^3 + 3x^2 + 3x + 1 - 3x - 6 - \frac{3}{x} + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$

$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}.$

6. $\{(a^{\frac{1}{4}} + b^{\frac{1}{4}}) - c^{\frac{1}{4}}\}^3 = (a^{\frac{1}{4}} + b^{\frac{1}{4}})^3 - 3(a^{\frac{1}{4}} + b^{\frac{1}{4}})^2.c^{\frac{1}{4}} + 3(a^{\frac{1}{4}} + b^{\frac{1}{4}})c^{\frac{1}{2}} - c^{\frac{3}{4}}$

$= a^{\frac{3}{4}} + 3a^{\frac{1}{2}}b^{\frac{1}{4}} + 3a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}} - 3a^{\frac{1}{2}}c^{\frac{1}{4}} - 6a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}} - 3b^{\frac{1}{2}}c^{\frac{1}{4}} + 3a^{\frac{1}{4}}c^{\frac{1}{2}}$

$+ 3b^{\frac{1}{4}}c^{\frac{1}{2}} - c^{\frac{3}{4}}.$

CLI.

1. The 8th term of $(1+x)^{11}$ is $\dfrac{11.10.9.8.7.6.5}{1.2.3.4.5.6.7}\cdot x^7 = 330x^7$.

2. The 5th term of $(a^2-b^2)^{12}$ is $\dfrac{12.11.10.9}{1.2.3.4}\cdot(a^2)^{12-5+1}.(b^2)^4 = 495a^{16}.b^8$.

3. The 4th term of $(a-b)^{100}$ is $-\dfrac{100.99.98}{1.2.3}\cdot a^{100-4+1}.b^3 = -161700a^{97}b^3$.

4. The 9th term of $(2ab-cd)^{14}$ is $\dfrac{14.13.12.11.10.9.8.7}{1.2.3.4.5.6.7.8}\cdot(2ab)^{14-9+1}.(cd)^8$

   $=192192a^6b^6c^8d^8$.

5. The 9th term of $(a-b)^{16}$ is $\dfrac{16.15.14.13.12.11.10.9}{1.2.3.4.5.6.7.8}\cdot a^{16-9+1}.b^8$

   $=12870a^8b^8$.

6. The 5th term of $(a^{\frac{1}{3}}+b^{\frac{1}{3}})^8$ is $\dfrac{8.7.6.5}{1.2.3.4}\cdot(a^{\frac{1}{3}})^{8-5+1}.(b^{\frac{1}{3}})^4 = 70a^{\frac{4}{3}}b^{\frac{4}{3}}$.

7. The 10th term of $(a-b)^{19}$ is $-\dfrac{19.18.17.16.15.14.13.12.11}{1.2.3.4.5.6.7.8.9}\cdot a^{10}.b^9$

   $=-92378a^{10}b^9$.

   The 11th term of $(a-b)^{19}$ is therefore $92378a^9b^{10}$.

8. The 7th term of $(a+x)^{13}$ is $\dfrac{13.12.11.10.9.8}{1.2.3.4.5.6}\cdot a^7.x^6 = 1716a^7x^6$.

   The 8th term of $(a+x)^{13}$ is therefore $1716a^6x^7$.

9. The coefficient of the rth term of $(a+x)^n$ is $\dfrac{n.(n-1)\ldots(n-r+2)}{1.2\ .\ .\ .\ (r-1)}$,

   and the middle term of $(a+x)^{4n}$ is the $(2n+1)$th.

$$\therefore \text{ Coefficient } = \frac{4n.(4n-1)\ .\ .\ .\ (4n-2n-1+2)}{1.2\ .\ .\ .\ (2n+1-1)}$$

$$= \frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}\cdot\frac{2n.(2n-1)\ .\ .\ .\ 1}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-2)\ .\ .\ .\ 6.4.2}{2n.(2n-1)\ .\ .\ .\ 3.2.1}\cdot\frac{(4n-1)(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2.2\ .\ .\ .\ \text{to } 2n \text{ factors}.\ \frac{(4n-1).(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2^{2n}.\ \frac{1.3.5\ .\ .\ .\ (4n-1)}{1.2.3\ .\ .\ .\ 2n}.$$

10. The middle term is the $(2n+2)$th.

$$\therefore \text{ Coefficient } =\frac{(4n+2)(4n+1)\ .\ .\ .\ (4n+2-2n-2+2)}{1.2\ .\ .\ .\ (2n+2-1)}$$

$$=\frac{(4n+2).(4n+1)\ .\ .\ .\ (2n+2)}{1.2\ .\ .\ .\ (2n+1)}$$

$$=\frac{(4n+2)(4n)(4n-2)...(2n+4)(2n+2)}{(2n+1)(2n)(2n-1)...(n+1)}\cdot\frac{(4n+1)(4n-1)...(2n+3)}{n.(n-1)...2.1}$$

$$=2.2\ .\ .\ .\ \text{to } (n+1) \text{ factors}.\ \frac{(4n+1)(4n-1)\ .\ .\ .\ (2n+3)}{n.(n-1)\ .\ .\ .\ 2.1}$$

$$=2^{n+1}.\frac{(2n+3)(2n+5)\ .\ .\ .\ (4n-1)(4n+1)}{1.2\ .\ .\ .\ n}.$$

CLII.

1. $$1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}x^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^4$$

$$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4.$$

2. $$1+\frac{2}{3}a+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}a^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1.2.3}a^3$$

$$=1+\frac{2a}{3}-\frac{a^2}{9}+\frac{4a^3}{81}.$$

3. $$a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{a}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\cdot\frac{x^2}{a^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\cdot\frac{x^3}{a^3}\right.$$

$$\left.+\frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{1.2.3.4}\cdot\frac{x^4}{a^4}\right\}$$

$$=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{x^2}{9a^{\frac{5}{3}}}+\frac{5x^3}{81a^{\frac{8}{3}}}-\frac{10x^4}{243a^{\frac{11}{3}}}.$$

4. $$1+x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}(2x)^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}(2x)^4$$

$$=1+x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{5x^4}{8}.$$

5. $$a^{\frac{3}{4}}\left\{1+\frac{x}{a}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{4x}{3a}\right)^2+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{4x}{3a}\right)^3\right\}$$

$$=a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{1}{6}x^{-\frac{5}{4}}x^2+\frac{5}{54}a^{-\frac{9}{4}}.x^3.$$

6. $$a^{\frac{1}{5}}\left\{1+\frac{4}{5}\cdot\frac{x^{\frac{1}{4}}}{a^{\frac{1}{4}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)}{1.2}\cdot\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)\cdot\left(\frac{4}{5}-2\right)}{1.2.3}\cdot\frac{x^{\frac{3}{4}}}{a^{\frac{3}{4}}}\right\}$$

$$=a^{\frac{1}{5}}+\frac{4}{5}\cdot a^{-\frac{1}{20}}.x^{\frac{1}{4}}-\frac{2}{25}a^{-\frac{3}{10}}.x^{\frac{1}{2}}+\frac{4}{125}\cdot a^{-\frac{11}{20}}.x^{\frac{3}{4}}.$$

7. $$1-\frac{x^2}{2}+\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{1.2}\cdot x^4-\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}-2\right)}{1.2.3}x^6$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^8$$

$$=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}.$$

8. $$1-\frac{7a^2}{3}+\frac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)}{1.2}a^4-\frac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{1.2.3}a^6$$

$$=1-\frac{7a^2}{3}+\frac{14a^4}{9}-\frac{14a^6}{81}.$$

9. $$1-\frac{9x}{4}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}(3x)^2-\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot(3x)^3$$

$$=1-\frac{9x}{4}-\frac{27x^2}{32}-\frac{135x^3}{128}.$$

10. $$x^3\left\{1-\frac{3}{2}\cdot\frac{2y}{3x^2}+\frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)}{1.2}\cdot\frac{4y^2}{9x^4}-\frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1.2.3}\cdot\frac{8y^3}{27x^6}\right\}$$

$$=x^3-xy+\frac{y^2}{6x}+\frac{y^3}{54x^3}.$$

11. $$1-\frac{5}{6}x+\frac{\frac{5}{6}\cdot\left(\frac{5}{6}-1\right)}{1.2}x^2-\frac{\frac{5}{6}\left(\frac{5}{6}-1\right)\left(\frac{5}{6}-2\right)}{1.2.3}x^3$$

$$=1-\frac{5}{6}x-\frac{5}{72}x^2-\frac{35}{1296}x^3.$$

12. $$\left(\frac{2x}{3}\right)^{\frac{2}{3}}\left\{1-\frac{2}{3}\cdot\frac{9y}{4x}+\frac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)}{1.2}\cdot\frac{81y^2}{16x^2}\right\}$$

$$=\left(\frac{2}{3}\right)^{\frac{2}{3}}.x^{\frac{2}{3}}-\left(\frac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{1}{3}}y-\frac{3}{8}\left(\frac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{4}{3}}y^2.$$

CLIII.

1. $1-2a+\dfrac{(-2)(-2-1)}{1.2}a^2+\dfrac{(-2)(-2-1)(-2-2)}{1.2.3}a^3+$

$$\frac{(-2)(-2-1)(-2-2)(-2-3)}{1.2.3.4}a^4=1-2a+3a^2-4a^3+5a^4.$$

2. $1+3x+\dfrac{(-1)\cdot(-2)}{1.2}(3x)^2-\dfrac{(-1).(-2).(-3)}{1.2.3}(3x)^3+$

$$\frac{(-1).(-2).(-3).(-4)}{1.2.3.4}(3x)^4=1+3x+9x^2+27x^3+81x^4.$$

3. $1+x+\dfrac{(-4).(-5)}{1.2}\cdot\dfrac{x^2}{16}-\dfrac{(-4).(-5).(-6)}{1.2.3}\cdot\dfrac{x^3}{64}$

$$=1+x+\frac{5x^2}{8}+\frac{5x^3}{16}.$$

4. $1+x+\dfrac{(-2).(-3)}{1.2}\cdot\dfrac{x^2}{4}-\dfrac{(-2).(-3).(-4)}{1.2.3}\cdot\dfrac{x^3}{8}+$

$$\frac{(-2).(-3).(-4).(-5)}{1.2.3.4}\cdot\frac{x^4}{16}=1+x+\frac{3x^2}{4}+\frac{x^3}{2}+\frac{5x^4}{16}.$$

5. $a^{-10}-(-5)a^{-12}.2x+\dfrac{(-5)(-6)}{1.2}a^{-14}.4x^2-\dfrac{(-5).(-6)(-7)}{1.2.3}.a^{-16}.8x^3$

$$+\frac{(-5).(-6).(-7).(-8)}{1.2.3.4}a^{-18}.16x^4$$

$$=a^{-10}+10a^{-12}x+60a^{-14}x^2+280a^{-16}.x^3+1120a^{-18}x^4.$$

6. $a^{-2}-(-6)a^{-\frac{7}{3}}\cdot x^{\frac{1}{3}}+\dfrac{(-6)\cdot(-7)}{1.2}\cdot a^{-\frac{8}{3}}\cdot x^{\frac{2}{3}}-\dfrac{(-6)\cdot(-7)\cdot(-8)}{1.2.3}\cdot a^{-3}x$

$$=\frac{1}{a^2}+\frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}}+\frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}}+\frac{56x}{a^3}.$$

CLIV.

1. $1-\dfrac{1}{2}x^2+\dfrac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}x^4+\dfrac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}x^6$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}x^8$$

$$=1-\frac{x^2}{2}+\frac{3x^4}{8}-\frac{5x^6}{16}+\frac{35x^8}{128}.$$

2. $$1+\frac{3}{2}x^2+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2}x^4-\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3}x^6$$

$$+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)\cdot\left(-\frac{9}{2}\right)}{1.2.3.4}x^8$$

$$=1+\frac{3x^2}{2}+\frac{15x^4}{8}+\frac{35x^6}{16}+\frac{315x^8}{128}.$$

3. $$x^{-2}\left\{1-\frac{2}{5}\cdot\frac{z^5}{x^5}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)}{1.2}\cdot\frac{z^{10}}{x^{10}}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)\cdot\left(-\frac{12}{5}\right)}{1.2.3}\frac{z^{15}}{x^{15}}\right\}$$

$$=x^{-2}-\frac{2}{5}x^{-7}z^5+\frac{7}{25}x^{-12}z^{10}-\frac{28}{125}x^{-17}.z^{15}.$$

4. $$1-x+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}4x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}8x^3$$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}16x^4$$

$$=1-x+\frac{3x^2}{2}-\frac{5x^3}{2}+\frac{35x^4}{8}.$$

5. $$a^{-1}\left\{1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}\cdot\frac{x^4}{a^4}\right.$$

$$\left.+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}\cdot\frac{x^6}{a^6}\right\}$$

$$=\frac{1}{a}-\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}-\frac{5x^6}{16a^7}.$$

6. $$a^{-1}\left\{1-\frac{1}{3}\cdot\frac{x^3}{a^3}+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)}{1.2}\cdot\frac{x^6}{a^6}\right.$$

$$\left.+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)}{1.2.3}\cdot\frac{x^9}{a^9}\right\}=\frac{1}{a}-\frac{x^3}{3a^4}+\frac{2x^6}{9a^7}-\frac{14x^9}{81a^{10}}.$$

CLV.

The rth term of $(a+x)^n$ is $\frac{n.(n-1)\ .\ .\ .\ (n-r+2)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}.\ a^{n-r+1}.$

1. The rth term is $\frac{7\cdot 6\ .\ .\ .\ (9-r)}{1\cdot 2\ .\ .\ .\ (r-1)}x^{r-1}.$

2. The rth term is $(-1)^{r-1}\cdot\frac{12.11\ .\ .\ .\ (14-r)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}.$

3. The rth term is $(-1)^{r-1}\cdot\frac{8.7\ .\ .\ .\ (10-r)}{1.2\ .\ .\ .\ (r-1)}a^{9-r}\cdot x^{r-1}.$

4. The rth term is $\frac{9.8\ .\ .\ .\ (11-r)}{1.2\ .\ .\ .\ (r-1)}\cdot(5x)^{10-r}\cdot(2y)^{r-1}.$

5. The rth term is $\frac{(-2)\cdot(-3)\ .\ .\ .\ (-2-r+2)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}.$

$$=(-1)^{r-1}\cdot\frac{2.3\ .\ .\ .\ r}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}=(-1)^{r-1}\cdot r\cdot x^{r-1}.$$

6. The rth term is $\frac{4.5\ .\ .\ .\ (r+2)}{1.2\ .\ .\ .\ (r-1)}\cdot(3x)^{r-1}$

$$=\frac{r\cdot(r+1)\cdot(r+2)}{1.2.3}\cdot(3x)^{r-1}.$$

7. The rth term is $\frac{\frac{1}{2}\cdot\frac{3}{2}\ .\ .\ .\ \left(r-\frac{3}{2}\right)}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}$

$$=\frac{\left\{1.3.5 \ldots (2r-3)\right\} \times \left(\frac{1}{2}\right)^{r-1}}{1.2 \ldots (r-1)} \cdot x^{r-1}$$

$$=\frac{1.3.5 \ldots (2r-3)}{1.2 \ldots (r-1)} \cdot \left(\frac{x}{2}\right)^{r-1}.$$

8. The rth term is $(-1)^{r-1} \cdot \dfrac{\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \ldots \left(r-\frac{7}{3}\right)}{1.2 \ldots (r-1)} \cdot a^{\frac{1}{3}-r+1}x^{r-1}$

$$=(-1)^{r-1} \cdot \frac{\left\{1.2.5 \ldots (3r-7)\right\} \times \left(\frac{1}{3}\right)^{r-1}}{1.2 \ldots (r-1)} \cdot a^{\frac{1}{3}} \cdot \left(\frac{x}{a}\right)^{r-1}$$

$$=(-1)^{r-1} \cdot \frac{1.2.5 \ldots (3r-7)}{1.2 \ldots (r-1)} \cdot a^{\frac{1}{3}} \cdot \left(\frac{x}{3a}\right)^{r-1}$$

$$=\frac{1.2.5 \ldots (3r-7)}{1.2 \ldots (r-1)} \cdot a^{\frac{1}{3}} \cdot \left(-\frac{x}{3a}\right)^{r-1}.$$

9. The rth term is $\dfrac{\frac{7}{2} \cdot \left(\frac{7}{2}+1\right) \ldots \left(\frac{7}{2}+r-2\right)}{1.2 \ldots (r-1)} \cdot (2x)^{r-1}$

$$=\frac{\left\{7.9 \ldots (2r+3)\right\} \times \left(\frac{1}{2}\right)^{r-1}}{1.2 \ldots (r-1)} \cdot (2x)^{r-1}$$

$$=\frac{7.9 \ldots (2r+3)}{1.2 \ldots (r-1)} \cdot x^{r-1}.$$

10. The rth term is $\dfrac{\frac{3}{4} \cdot \left(\frac{3}{4}+1\right) \ldots \left(\frac{3}{4}+r-2\right)}{1.2 \ldots (r-1)} . (a^2)^{-\frac{3}{4}-r+1} . (x^2)^{r-1}$

$$=\frac{\{3.7 \ldots (4r-5)\} \times \left(\frac{1}{4}\right)^{r-1}}{1.2 \ldots (r-1)} . a^{-\frac{3}{2}} . \left(\frac{x^2}{a^2}\right)^{r-1}$$

$$=\frac{3.7 \ldots (4r-5)}{1.2 \ldots (r-1)} \cdot \frac{a^{-\frac{3}{2}}}{4^{r-1}} \cdot \left(\frac{x}{a}\right)^{2(r-1)}$$

11. The $(r+1)$th term is $\dfrac{3.4\ .\ .\ .\ (3+r-1)}{1.2\ .\ .\ .\ r}x^r$

$$=\frac{3.4\ .\ .\ .\ (2+r)}{1.2\ .\ .\ .\ r}.x^r=\frac{(r+1)(r+2)}{2}.x^r.$$

2. The $(r+1)$th term is $\dfrac{\frac{1}{2}\left(\frac{1}{2}+1\right)\ .\ .\ .\ \left(\frac{1}{2}+r-1\right)}{1.2\ .\ .\ .\ r}(4x)^r$

$$=\frac{\{1.3\ .\ .\ .\ (2r-1)\}\times\left(\frac{1}{2}\right)^r}{1.2\ .\ .\ .\ r}.(4x)^r$$

$$=\frac{1.3\ .\ .\ .\ (2r-1)}{1.2\ .\ .\ .\ r}.(2x)^r.$$

13. The $(r+1)$th term is $\dfrac{2r.(2r-1)\ .\ .\ .\ (r+1)}{1.2\ .\ .\ .\ r}x^r$

$$=\frac{2r.(2r-1)\ .\ .\ .\ (r+1).r.(r-1)\ .\ .\ .\ 2.1}{(\underline{|r})^2}.x^r$$

$$=\frac{(2r-1).(2r-3)\ .\ .\ .\ 3.1}{\underline{|r}}.\frac{2r.(2r-2)\ .\ .\ .\ 4.2}{\underline{|r}}.x^r$$

$$=\frac{(2r-1)(2r-3)\ .\ .\ .\ 3.1}{\underline{|r}}.\frac{2^r.(\underline{|r})}{\underline{|r}}.x^r$$

$$=\frac{1.3.5\ .\ .\ .\ (2r-1)}{1.2.3\ .\ .\ .\ r}.(2x)^r.$$

14. Coefficient of x^{r+1} in $(1+x)^{n+1}$ is $\dfrac{(n+1.)n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ (r+1)}$

Coefficient of x^r in $(1+x)^n$ is $\dfrac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ r}$.

Coefficient of x^{r+1} in $(1+x)^n$ is $\dfrac{n.(n-1)\ .\ .\ .\ (n-r)}{1.2\ .\ .\ .\ (r+1)}$.

Now $\dfrac{(n+1)n(n-1)\ldots(n-r+1)}{1.2.3\ .\ .\ .\ (r+1)}=\dfrac{n.(n-1)\ldots(n-r+1)}{1.2\ .\ .\ .\ r}.\dfrac{n+1}{r+1}$

$$=\frac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}.\left(1+\frac{n-r}{r+1}\right)$$

$$=\frac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}+\frac{n.(n-1)\ .\ .\ .\ (n-r)}{1.2\ .\ .\ .\ (r+1)}.$$

15. $\dfrac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}.(a)^{-\frac{1}{2}-3}.\left(\dfrac{1}{x}\right)^3=\dfrac{5}{16}a^{-\frac{7}{2}}.\dfrac{1}{x^3}.$

16. $\dfrac{\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2.3.4}.(a^2)^{\frac{3}{2}-4}.(-b^2)^4=\dfrac{3}{128}a^{-5}.b^8.$

17. $-\dfrac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{1.2.3.4.5.6.7.8}.(a^2)^{\frac{1}{2}-8}.(2x^2)^8$

$=-\dfrac{429}{128}.a^{-15}.x^{16}.$

18. $-\dfrac{m.(m+1)\ .\ .\ .\ (m+8)}{1.2\ .\ .\ .\ 9}.a^{-(m+9)}.b^9.$

19. $\dfrac{\frac{1}{m}\cdot\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\ .\ .\ .\ \left(\frac{1}{m}-5\right)}{1.2\ .\ .\ .\ 6}a^{\frac{1}{m}-6}b^6$

$=\dfrac{(1-m)(1-2m)\ .\ .\ .\ (1-5m)}{1.2\ .\ .\ .\ 6.m^6}.a^{\frac{1}{m}-6}.b^6.$

CLVI.

1. $\sqrt[3]{31}=\sqrt[3]{(27+4)}=3\left(1+\dfrac{4}{27}\right)^{\frac{1}{3}}$

$=3\left\{1+\dfrac{1}{3}\cdot\dfrac{4}{27}+\dfrac{1}{2}\cdot\dfrac{1}{3}\cdot\left(\dfrac{1}{3}-1\right)\cdot\dfrac{16}{729}\right.$

$+\dfrac{1}{6}\cdot\dfrac{1}{3}\cdot\left(\dfrac{1}{3}-1\right)\cdot\left(\dfrac{1}{3}-2\right)\cdot\dfrac{64}{19683}$

$\left.+\dfrac{1}{24}\cdot\dfrac{1}{3}\cdot\left(\dfrac{1}{3}-1\right)\cdot\left(\dfrac{1}{3}-2\right)\cdot\left(\dfrac{1}{3}-3\right).\dfrac{256}{531441}+\ .\ .\ .\ .\right\}$

$=3+\dfrac{4}{27}-\dfrac{16}{2187}+\dfrac{320}{531441}-\dfrac{2560}{43046721}+\ .\ .\ .\ .$

$=3+\dfrac{6085724}{43046721}=3{\cdot}1413749\ .\ .\ .\ .$

2. $\sqrt[7]{108}=\sqrt[7]{(128-20)}=2\left(1-\frac{5}{32}\right)^{\frac{1}{7}}$

$$=2\left\{1-\frac{1}{7}\cdot\frac{5}{32}+\frac{1}{2}\cdot\frac{1}{7}\left(\frac{1}{7}-1\right)\frac{25}{1024}\right.$$

$$-\frac{1}{6}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\left(\frac{1}{7}-2\right)\cdot\frac{125}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\cdot\left(\frac{1}{7}-2\right)\cdot\left(\frac{1}{7}-3\right)\cdot\frac{625}{1048576}-\ldots\right\}$$

$$=2\left\{1-\frac{5}{224}-\frac{75}{50176}-\frac{1625}{11239424}-\frac{40625}{2517630976}\ldots\right\}$$

$$=2\left\{1-\frac{56197120+3763200+364000+40625}{2517630976}\right\}$$

$$=2\left(1-\frac{60364945}{2517630976}\right)=\frac{2457266031}{1258815488}=1\cdot95204\ldots$$

3. $\sqrt[5]{260}=\sqrt[5]{(243+17)}=3\left(1+\frac{17}{243}\right)^{\frac{1}{5}}$

$$=3\left\{1+\frac{1}{5}\cdot\frac{17}{243}+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{289}{59049}\right.$$

$$\left.+\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{4913}{14348907}\right\}$$

$$=3.\left\{1+\frac{17}{1215}-\frac{578}{1476225}+\frac{29478}{1793613375}\ldots\right\}$$

$$=3\left\{1+\frac{25095825-702270+29478}{1793613375}\ldots\right\}$$

$$=3.\left\{1+\frac{24423033}{1793613375}\right\}=3+\frac{24423033}{597871125}=3\cdot04084\ldots$$

4. $\sqrt[5]{31}=\sqrt[5]{(32-1)}=2\left(1-\frac{1}{32}\right)^{\frac{1}{5}}=2\left\{1-\frac{1}{5}\cdot\frac{1}{32}\right.$

$$+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{1}{1024}-\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{1}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\left(\frac{1}{5}-3\right)\cdot\frac{1}{1048576}\cdots\right\}$$

$$=2\left\{1-\frac{1}{160}-\frac{1}{12800}-\frac{3}{4096000}-\frac{21}{655360000}\dots\right\}$$

$$=2\left\{1-\frac{4096000+51200+480+21}{655360000}\dots\right\}$$

$$=2\left\{1-\frac{4147701}{655360000}\right\}=1{\cdot}98734\dots$$

CLVII.

1.
```
  23561
  42513
 645325
-------
1045032
```

2.
```
 3074852
 4635628
 1247153
--------
10070344
```

3.
```
358423
267862
------
 80451
```

4.
```
211010
124321
------
 31134
```

5.
```
   57264
     675
   ------
  354604
 513354
434070
--------
51117344
```

6.
```
    1456
    6451
    ----
    1456
   6523
 11312
13101
--------
14332216
```

7.
```
5)243012
  ------
  31450 rem. 2.
```

8.
```
6)3756025
  -------
  522256 rem. 1.
```

9.
```
        25|40|05|44(4112
        24
        ---
121      140
         121
         ----
1221     1505
         1221
         -----
12222    24444
         24444
         -----
```

10.
```
      5|68|98|t1(2437
      4
      ---
44    168
      154
      ----
483   1498
      1209
      -----
4867  28et1
      28et1
      -----
```

CLVIII.

1.
$$\begin{array}{r|l} 7 & 1828 \\ \hline 7 & 261-1 \\ \hline 7 & 37-2 \\ \hline 7 & 5-2 \\ \hline & 0-5 \end{array}$$

2.
$$\begin{array}{r|l} 6 & 1820 \\ \hline 6 & 303-2 \\ \hline 6 & 50-3 \\ \hline 6 & 8-2 \\ \hline 6 & 1-2 \\ \hline & 0-1 \end{array}$$

3.
$$\begin{array}{r|l} 12 & 43751 \\ \hline 12 & 3645-e \\ \hline 12 & 303-9 \\ \hline 12 & 25-3 \\ \hline 12 & 2-1 \\ \hline & 0-2 \end{array}$$

4.
$$\begin{array}{r|l} 5 & 3700 \\ \hline 5 & 740-0 \\ \hline 5 & 148-0 \\ \hline 5 & 29-3 \\ \hline 5 & 5-4 \\ \hline 5 & 1-0 \\ \hline & 0-1 \end{array}$$

5.
$$\begin{array}{r|l} 2 & 7631 \\ \hline 2 & 3815-1 \\ \hline 2 & 1907-1 \\ \hline 2 & 953-1 \\ \hline 2 & 476-1 \\ \hline 2 & 238-0 \\ \hline 2 & 119-0 \\ \hline 2 & 59-1 \\ \hline 2 & 29-1 \\ \hline 2 & 14-1 \\ \hline 2 & 7-0 \\ \hline 2 & 3-1 \\ \hline 2 & 1-1 \\ \hline & 0-1 \end{array}$$

6.
$$\begin{array}{r|l} 12 & 215855 \\ \hline 12 & 17987-e \\ \hline 12 & 1498-e \\ \hline 12 & 124-t \\ \hline 12 & 10-4 \\ \hline & 0-t \end{array}$$

7.
$$\begin{array}{r|l} 7 & 790158 \\ \hline 7 & 112879-5 \\ \hline 7 & 16125-4 \\ \hline 7 & 2303-4 \\ \hline 7 & 329-0 \\ \hline 7 & 47-0 \\ \hline 7 & 6-5 \\ \hline & 0-6 \end{array}$$

8.
$$\begin{array}{r|l} 4 & 34002 \\ \hline 4 & 4334-1 \\ \hline 4 & 1043-2 \\ \hline 4 & 122-0 \\ \hline 4 & 14-1 \\ \hline 4 & 2-1 \\ \hline & 0-2 \end{array}$$

9. $$\begin{array}{r|l} 12 & 8978 \\ 12 & 816-2 \\ 12 & 75-1 \\ 12 & 6-t \\ & 0-6 \end{array}$$

10. $$\begin{array}{r|l} 12 & 3256 \\ 12 & 166-4 \\ 12 & 11-1 \\ & 0-8 \end{array}$$

11. $$\begin{array}{r|l} 8 & 37704 \\ 8 & 4311-5 \\ 8 & 480-1 \\ 8 & 54-4 \\ 8 & 6-1 \\ & 0-6 \end{array}$$

2. $$\begin{array}{r|l} 4 & 5056 \\ 4 & 1165-0 \\ 4 & 214-3 \\ 4 & 36-1 \\ 4 & 6-3 \\ 4 & 1-2 \\ & 0-1 \end{array}$$

13. $$\begin{array}{r|l} 7 & 654321 \\ 7 & e0738-5 \\ 7 & 16e3e-3 \\ 7 & 2858-3 \\ 7 & 478-0 \\ 7 & 7e-3 \\ 7 & 11-4 \\ 7 & 1-6 \\ & 0-1 \end{array}$$

14. $$\begin{array}{r|l} 11 & 2304 \\ 11 & 104-t \\ 11 & 2-7 \\ & 0-2 \end{array}$$

CLIX.

1. $$\left.\begin{array}{l} \frac{25}{36}\times 6=4+\frac{6}{36} \\ \frac{6}{36}\times 6=1+0 \end{array}\right\} \therefore \cdot 41 \text{ is the result.}$$

2. $\frac{3}{11}\times 7=1+\frac{10}{11}$; $\frac{10}{11}\times 7=6+\frac{4}{11}$; $\frac{4}{11}\times 7=2+\frac{6}{11}$;

$\frac{6}{11}\times 7=3+\frac{9}{11}$; $\frac{9}{11}\times 7=5+\frac{8}{11}$; $\frac{8}{-}\times 7=5+\frac{1}{11}$;

$\frac{1}{11}\times 7=1+\frac{7}{11}$; $\frac{7}{11}\times 7=4+\frac{5}{11}$; $\frac{5}{11}\times 7=3+\frac{2}{11}$;

$\frac{2}{11}\times 7=1+\frac{3}{11}$; $\frac{3}{11}\times 7=1+\frac{10}{11}$

$\therefore$ result is $\cdot\dot{1}6235504\dot{3}$.

3.
$$\begin{array}{r|l} 9 & 23 \\ \hline 9 & 2-5 \\ \hline & 0-2 \end{array} \qquad \begin{array}{r} \cdot 125 \\ 9 \\ \hline 1\cdot 125 \\ 9 \\ \hline 1\cdot 125 \end{array}$$

$\therefore$ result is $25\cdot\dot{1}$.

4.
$$\begin{array}{r|l} 6 & 1820 \\ \hline 6 & 303-2 \\ \hline 6 & 50-3 \\ \hline 6 & 8-2 \\ \hline 6 & 1-2 \\ \hline & 0-1 \end{array} \qquad \begin{array}{r} \cdot 3375 \\ 6 \\ \hline 2\cdot 0250 \\ 6 \\ \hline 0\cdot 1500 \\ 6 \\ \hline 0\cdot 9000 \\ 6 \\ \hline 5\cdot 4000 \\ 6 \\ \hline 2\cdot 4000 \end{array}$$

$\therefore$ result is $12232\cdot 2005\dot{2}$.

5. $2x^5+x^4+2x^3+5x^2+4x+2-17486=0$;

$(x-6)(2x^4+13x^3+80x^2+485x+2914)=0$;

$\therefore x=6$. Hence the scale is senary.

6. $x^6+7x^5+4x^4+6x^3+3x^2+5-511173=0$;

$(x-8)(x^5+15x^4+124x^3+998x^2+7987x+63896)=0$;

$\therefore x=8$. Hence the scale is octenary.

7. (1.) Let N be the number, and suppose

$N = a.10^n + b.10^{n-1} + \ \ldots \ + m.100 + p.10 + q.$

Then $N = a(10^n - 1) + b(10^{n-1} - 1) + \ \ldots \ + m(100 - 1)$
$+ p(10 - 1) + (a + b + \ \ldots \ + m + p + q).$

Now all the expressions $10^n - 1, 10^{n-1} - 1 \ldots 10^2 - 1, 10 - 1$ are divisible by $10 - 1$, or 9, and therefore by 3.

Hence N is divisible by 3, if $a + b + \ \ldots \ + m + p + q$ be divisible by 3.

(2.) Let $N = a.10^n + b.10^{n-1} + \ \ldots \ + m.100 + p.10 + q.$

Now 100 and all its multiples are divisible by 4,

$\therefore N$ is divisible by 4 if $10p + q$ be divisible by 4.

(3.) Let $N = a.10^n + b.10^{n-1} + \ \ldots \ + m.100 + p.10 + q.$

Now 1000 and all its multiples are divisible by 8,

$\therefore N$ is divisible by 8 if $100m + 10p + q$ be divisible by 8.

(4.) Let $N = a.10^n + b.10^{n-1} + \ \ldots \ + m.100 + p.10 + q.$

Now 10 and all its multiples are divisible by 5,

$\therefore N$ is divisible by 5 if $q = 5$ or $q = 0$.

(5.) Let N be the number, $p_n, p_{n-1}, \ldots p_4, p_3, p_2, p_1$, the digits.

Then $N = p_1 + 10.p_2 + 100p_3 + 1000p_4 + \ldots + 10^{n-1}.p_{n-1} + 10^n p^n$
$= p_1 - p_2 + p_3 - p_4 + \ \ldots \ + (-1)^n.p_n$
$+ p_2(10 + 1) + p_3(10^2 - 1) + p_4(10^3 + 1) + \ldots + p_n\{10^n - (-1)^n\}.$

Now $10 + 1, 10^2 - 1 \ \ldots$ are all divisible by $10 + 1$, or 11,

$\therefore N$ is divisible by 11, if $(p_1 + p_3 + \ \ldots \ ) - (p_2 + p_4 \ \ldots)$ be divisible by 11.

8. Let $N = a.r^n + b.r^{n-1} + \ \ldots \ + m.r^2 + pr + q,$
then $n = a + rb + \ \ldots \ + m.r^{n-2} + p.r^{n-1} + q.r^n.$

Then $N - n = a(r^n - 1) + b.(r^{n-1} - r) + \ \ldots \ - m(r^{n-2} - r^2)$
$- p(r^{n-1} - r) - q(r^n - 1)$, and each of the factors $r^n - 1$, $r^{n-1} - r \ \ldots \ r^{n-2} - r^2, r^{n-1} - r, r^n - 1$ is divisible by $r - 1$;

$\therefore N - n$ is divisible by $r - 1$.

CLX.

1. $$\begin{array}{r} \bar{3}\cdot1651553 \\ \bar{4}\cdot7505855 \\ 6\cdot6879746 \\ \bar{2}\cdot6150026 \\ \hline \bar{1}\cdot2187180 \end{array}$$

2. $$\begin{array}{r} \bar{4}\cdot6843785 \\ \bar{5}\cdot6650657 \\ 3\cdot8905196 \\ \bar{3}\cdot4675284 \\ \hline \bar{7}\cdot7074922 \end{array}$$

3. $$\begin{array}{r} 2\cdot5324716 \\ 3\cdot6650657 \\ \bar{5}\cdot8905196 \\ \cdot3156215 \\ \hline 2\cdot4036784 \end{array}$$

4. $$\begin{array}{r} 2\cdot483269 \\ \bar{3}\cdot742891 \\ \hline 4\cdot740378 \end{array}$$

5. $$\begin{array}{r} \bar{2}\cdot352678 \\ \bar{5}\cdot428619 \\ \hline 2\cdot924059 \end{array}$$

6. $$\begin{array}{r} \bar{5}\cdot349162 \\ \bar{3}\cdot624329 \\ \hline \bar{3}\cdot724833 \end{array}$$

7. $$\begin{array}{r} \bar{2}\cdot4596721 \\ 3 \\ \hline \bar{5}\cdot3790163 \end{array}$$

8. $$\begin{array}{r} \bar{7}\cdot429683 \\ 6 \\ \hline \overline{40}\cdot578098 \end{array}$$

9. $$\begin{array}{r} \bar{9}\cdot2843617 \\ 7 \\ \hline \overline{62}\cdot9905319 \end{array}$$

10. $$3\,\underline{)\,\bar{6}\cdot3725409} \qquad \bar{2}\cdot1241803$$

11. $$6\,\underline{)\,\overline{14}\cdot432962} \qquad \bar{3}\cdot738827$$

12. $$9\,\underline{)\,\bar{4}\cdot53627188} \qquad \bar{1}\cdot61514132$$

CLXI.

1. Log $128 = \log 2^7 = 7\log 2 = 2\cdot1072100$.

Log $125 = \log\frac{1000}{8} = \log 1000 - \log 8 = 3 - \log 2^3$

$= 3 - 3\log 2 = 3 - \cdot9030900 = 2\cdot0969100$.

Log $2500 = \log\frac{10000}{4} = \log 10000 - \log 4 = 4 - 2\log 2$

$= 4 - \cdot6020600 = 3\cdot3979400$.

2. Log $50 = \log\frac{100}{2} = \log 100 - \log 2 = 2 - \cdot3010300 = 1\cdot6989700$.

Log $\cdot005 = \log\frac{5}{1000} = \log 10 - \log 2 - 3 = -\log 2 - 2 = \bar{3}\cdot6989700$.

Log $196 = \log(49\times 4) = 2\log 7 + 2\log 2 = 2\cdot2922560$.

3. Log 6 = log 3 + log 2 = ·7781513.
Log 27 = 3 log 3 = 1·4313639.
Log 54 = log (27 × 2) = 3 log 3 + log 2 = 1·7323939.
Log 576 = log (9 × 64) = 2 log 3 + 6 log 2 = 2·7604226.

4. Log 60 = log (2 × 3 × 10) = log 2 + log 3 + log 10 = 1·7781513.

$\text{Log } {\cdot}03 = \log \frac{3}{100} = \log 3 - 2 = {\cdot}4771213 - 2 = \bar{2}{\cdot}4771213.$

$\text{Log } 1{\cdot}05 = \log \frac{105}{100} = \log \frac{21}{20} = \log 3 + \log 7 - \log 2 - 1 = {\cdot}0211893.$

$\text{Log } {\cdot}0000432 = \log \frac{16 \times 27}{10000000} = 4 \log 2 + 3 \log 3 - 7 = \bar{5}{\cdot}6354839.$

5. $\text{Log } {\cdot}00075 = \log 75 - 5 = \log 3 + \log 25 - 5 = \log \left(\frac{18}{2}\right)^{\frac{1}{2}} + \log 25 - 5$

$= \frac{1}{2}\{\log 18 - \log 2\} + \log 100 - \log 4 - 5.$

$= \frac{1}{2}\{1{\cdot}2552725 - {\cdot}3010300\} + 2 - {\cdot}6020600 - 5$

$= {\cdot}4771213 - {\cdot}6020600 - 3 = \bar{4}{\cdot}8750613.$

Log 31·5 = log (21 × 3 × 5) − 1 = log 21 + log 3 + 1 − log 2 − 1.

$= \log 21 + \frac{1}{2}\{\log 18 - \log 2\} - \log 2$

$= 1{\cdot}3222193 + {\cdot}4771212 - {\cdot}3010300 = 1{\cdot}4983105.$

6. $\text{Log } 2 = \log \frac{10}{5} = 1 - \log 5 = {\cdot}3010300$

$\text{Log } {\cdot}064 = \log. \frac{2^6}{1000} = 6 \log 2 - 3 = 6 - 6 \log 5 - 3 = \bar{2}{\cdot}8061800$

$\text{Log } \left\{\frac{2^{60}}{5^{20}}\right\}^{\frac{1}{14}} = \frac{1}{14}(60 \log 2 - 20 \log 5)$

$= \frac{1}{7}(30 - 30 \log. 5 - 10 \log 5) = \frac{1}{7}(30 - 27{\cdot}9588000)$

$= \frac{1}{7}(2{\cdot}0412000) = {\cdot}2916000.$

7. $\text{Log } 5 = \log \frac{10}{2} = 1 - \cdot3010300 = \cdot6989700$

$\log \cdot125 = \log \frac{5^3}{1000} = 3 \log 5 - 3 = 2\cdot0969100 - 3 = \bar{1}\cdot0969100$

$\text{Log } \left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}} = \log 5^{\frac{90}{15}} - \log 2^{\frac{40}{15}} = \log 5^6 - \log 2^{\frac{8}{3}}$

$= 6 \log 5 - \frac{8}{3} \log 2 = 6 \ (\log 10 - \log 2) - \frac{8}{3} \log 2$

$= 4\cdot1938200 - \cdot8027467 = 3\cdot3910733.$

8. $-2, 0, 2$; $1, 0, -1$.

9. 1593 is greater than 10^3 and less than 10^4 ; characteristic 3 ; 1593 is greater than 12^2 and less than 12^3 ; characteristic 2.

10. $\frac{4^{3y}}{2^{4y}} = 8$; $\frac{2^{6y}}{2^{4y}} = 2^3$; $2^{2y} = 2^3$; $2y = 3$, etc.

11. (*a*) $\log 2 = \frac{1}{2} \log 4 = \cdot3010300.$

$\text{Log } 25 = \log 100 - \log 4 = 2 - \cdot6020600 = 1\cdot3979400$

$\text{Log } 83\cdot2 = \log (80 \times 1\cdot04) = \frac{3}{2} \log 4 + \log 10 + \log 1\cdot04$

$= \cdot9030900 + 1 + \cdot0170333 = 1\cdot9201233.$

$\text{Log } (\cdot625)^{\frac{1}{100}} = \frac{1}{100} \left\{ \log 625 - \log 1000 \right\} = \frac{1}{100} \left\{ 2 \log 25 - 3 \right\}$

$= \frac{1}{100} \left\{ 2 \log 100 - 2 \log 4 - 3 \right\} = \frac{1}{100} \left\{ 4 - 1\cdot2041200 - 3 \right\}$

$= -\cdot0020412 = \bar{1}\cdot9979588.$

(*b*) $\text{Log } (1\cdot04)^{6000} = 6000 \log 1\cdot04 = 6000 \times \cdot0170333$

$= 102\cdot1998000$; $\therefore$ number of digits is 103.

12. (*a*) $\text{Log } 5 = \frac{1}{2} \log 25 = \cdot 6989700$

$\text{Log } 4 = 2 - \log 25 = \cdot 6020600$

$\text{Log } 51 \cdot 5 = \log 5 + \log 10 \cdot 3 = \cdot 6989700 + 1 \cdot 0128372 = 1 \cdot 7118072$

$\text{Log } (\cdot 064)^{\frac{1}{100}} = \frac{1}{100} \left\{ \log 64 - \log 1000 \right\} = \frac{1}{100} \left\{ 3 \log 4 - 3 \right\}$

$= \frac{1}{100} \left\{ 1 \cdot 8061800 - 3 \right\} = - \cdot 0119382 = \bar{1} \cdot 9880618.$

(*b*) $\text{Log } (1 \cdot 03)^{600} = 600 \log 1 \cdot 03 = 600 \times \cdot 0128372$

$= 7 \cdot 7023200$; $\therefore$ number of digits is 8.

13. $\text{Log } 7623 = 2 \log 3 + 2 \log 11 + \log 7$

$= \cdot 9542426 + 2 \cdot 0827854 + \cdot 8450980 = 3 \cdot 8821260$

$\text{Log } \frac{77}{300} = \log 7 + \log 11 - \log 3 - \log 100$

$= \cdot 8450980 + 1 \cdot 0413927 - \cdot 4771213 - 2 = \bar{1} \cdot 4093694$

$\text{Log } \frac{3}{539} = \log 3 - \log 11 - 2 \log 7$

$= \cdot 4771213 - 1 \cdot 0413927 - 1 \cdot 6901960 = \bar{3} \cdot 7455326.$

14. (1.) $x \log 4096 = \log 8 - x \log 64$; $4x \log 8 = \log 8 - 2x \log 8$;

$4x = 1 - 2x$; $6x = 1$; $x = \frac{1}{6}$.

(2.) $(2 \cdot 5)^x = 6 \cdot 25 = (2 \cdot 5)^2$; $\therefore x = 2$.

(3.) $(ab)^x = m$; $x \log (ab) = \log m$

$\therefore x = \dfrac{\log m}{\log a + \log b}$.

(4.) $x(m \log a + 2 \log b) = \log c$, etc.

(5.) $3x \log a + (4 - x) \log b = (2x - 1) \log c$

$x(3 \log a - \log b - 2 \log c) = -4 \log b - \log c$, etc.

(6.) $x (\log a + m \log b) = \log c - 3x \log c$

$x (\log a + m \log b + 3 \log c) = \log c$, etc.

CLXII.

1. $P(1+r)^n=2P$; $(1+r)^n=2$; $\left(1+\frac{4}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 104-\log 100}=\frac{\cdot 3010300}{\cdot 0170333}=17\cdot 6\ \ldots$$

2. $P(1+r)^n=2P$; $\left(1+\frac{3}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 1\cdot 03}=\frac{\cdot 3010300}{\cdot 0128372}=23\cdot 4\ \ldots$$

3. $\left(1+\frac{10}{100}\right)^n=2$; $n=\frac{\log 2}{\log 1\cdot 1}=\frac{\cdot 3010300}{\cdot 0413927}=7\cdot 2725\ \ldots$

4. $\left(1+\frac{5}{100}\right)^n=3$; $n=\frac{\log 3}{\log 1\cdot 05}=\frac{\cdot 4771213}{\cdot 0211893}=22\cdot 5$ nearly.

5. $P(1+r)^n=2P$; $\therefore n=\frac{\log 2}{\log (1+r)}$

$P(1+2r)^m=2P$; $\therefore m=\frac{\log 2}{\log (1+2r)}$

$$\therefore \frac{m}{n}=\frac{\log (1+r)}{\log (1+2r)}.$$

Now $1+2r$ is less than $(1+r)^2$

$\therefore \frac{m}{n}$ is greater than $\frac{\log (1+r)}{\log (1+r)^2}$, or, $\frac{\log (1+r)}{2 \log (1+r)}$, or, $\frac{1}{2}$

6. $1000(1+r)^n=1800$; $\left(1+\frac{5}{100}\right)^n=1\cdot 8$

$\therefore n \log 1\cdot 05=\log 1\cdot 8$; $n=\frac{\cdot 2552725}{\cdot 0211893}=12$ nearly.

7. $P(1+r)^{2n}=2P$; $\left(1+\frac{3}{100}\right)^{2n}=2$

$$\therefore 2n=\frac{\log 2}{\log 1\cdot 03}=\frac{\cdot 3010300}{\cdot 0128372}=23\cdot 449\ \ldots$$

$\therefore n=11\cdot 724\ \ldots .$

FINIS.

3, WATERLOO PLACE, PALL MALL,
January, 1882.

Educational Works

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